Combinatorial Mesh Calculus (CMC): Lecture 14

Lectured by: Dr. Kiprian Berbatov[1]

Lecture Notes Compiled by: Muhammad Azeem[1]

Under the supervision of: Prof. Andrey P. Jivkov[1]

[1] Department of Mechanical and Aerospace Engineering, The University of Manchester, Oxford Road, Manchester M13 9PL, UK

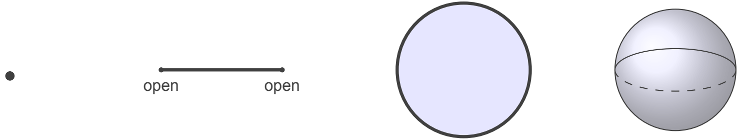
p–Cells

Definition (CRWU context implicit for smoothness)

Let $p \in \mathbb{N}$ and X be a set.

- If $p = 0$, then X is a single point: a 0–*cell*.
- If $p > 0$, then X is *diffeomorphic* to an open p–ball (open interval, disk, ball): a p–*cell*.

$p = 0$: point $p = 1$: open interval $p = 2$: open disk $p = 3$: open ball



Regular Cell Complex

Definition

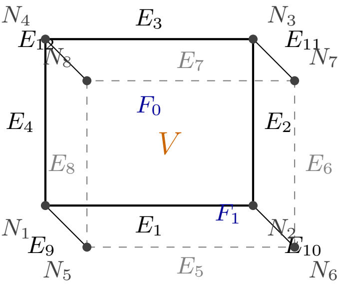
Let $D \in \mathbb{N}$, X a smooth D–manifold, and $\mathcal{M} \subseteq \mathcal{P}(X)$ a set of cells in X. We say $\mathcal{M}$ is a *regular cell complex* if:

(1) For any $a, b \in \mathcal{M}$ with $a \neq b$, either $a \cap b = \varnothing$ or $a \cap b$ is a union of cells of strictly lower dimension.

(2) For any $a \in \mathcal{M}$, its closure in X satisfies

$$\mathrm{cl}(a) = \bigcup \{\, b \in \mathcal{M} \mid b \subseteq a \,\}.$$

Example



The **closure operator** cl(σ^k) returns a complex containing the k-cell σ^k and all its boundary cells.

Example Closures:

Closure of Face F_0 (2-Cell):

cl$(F_0) = \{N_1, N_2, N_3, N_4\} \cup \{E_1, E_2, E_3, E_4\} \cup \{F_0\}$

Closure of Edge E_1 (1-Cell):

cl$(E_1) = \{N_1, N_2\} \cup \{E_1\}$

Faces and Partial Order

Remark (Face relation)

A regular cell complex $\mathcal{M}$ carries a partial order by inclusion: $a \subseteq b$. We write $a \preccurlyeq b$ and say "a is a *face* of b".

Hasse-type diagram (schematic)

$$
\begin{tikzcd}
 & & & V \arrow[dll, dash] \arrow[drr, dash] & & & \\
 & F_0 \arrow[dl, dash] \arrow[dr, dash] & & & & F_1 \arrow[dl, dash] \arrow[dr, dash] & \\
E_0 \arrow[d, dash] & & E_1 \arrow[dll, dash] \arrow[d, dash] & & E_2 \arrow[dll, dash] \arrow[d, dash] & & E_3 \arrow[dll, dash] \arrow[d, dash] \\
N_0 & & N_1 & & N_2 & & N_3
\end{tikzcd}
$$

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Combinatorial Mesh

Definition

A partially ordered set $(\mathcal{M}, \preccurlyeq)$ is a *combinatorial mesh* if there exists a manifold X and an embedding $\varphi : \mathcal{M} \to \mathcal{P}(X)$ whose image is a regular cell complex in X.

Relational data (example)

$$F_0 \prec E_0, E_1,$$
$$F_1 \prec E_2, E_3,$$
$$E_0 \prec N_0, \quad E_1 \prec N_0, N_1,$$
$$E_2 \prec N_1, N_2, \quad E_3 \prec N_2.$$

Example

Abstract Cell Complex M

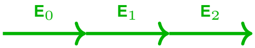
(Cells defined by incidence and dimension)

Geometric Realization $\varphi(\mathbf{M}) \subset \mathbb{R}^n$

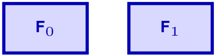
(Mapping the cells to space)

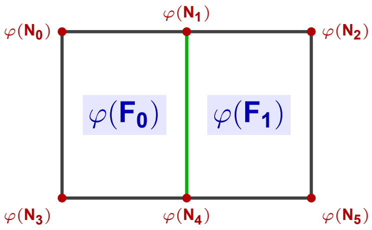
0-Cells ($\mathbf{N}_i$)

1-Cells ($\mathbf{E}_i$)



2-Cells ($\mathbf{F}_i$)





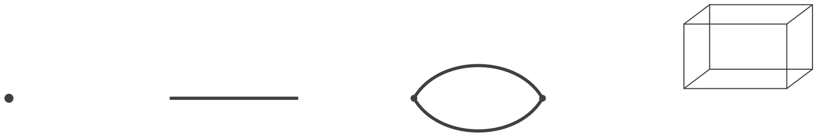
$\varphi : \mathbf{M} \to \mathbb{R}^{\mathbf{n}}$ (Map/Embedding)

Combinatorial Polytopes and Digon

Remark

If $\mathcal{M}$ is a combinatorial mesh, the set of p–cells is $\mathcal{M}_p$. If D is the maximum dimension, then $\mathcal{M} = \bigcup_{p=0}^{D} \mathcal{M}_p$. If there exists a single top D–cell containing all cells, the mesh is a *combinatorial polytope*.

0–polytope | 1–polytope | 2–polytope (digon) | 3–polytope (cube)



Diamond Property

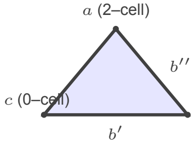
Proposition

Let $\mathcal{M}$ be a combinatorial mesh, $p \in \mathbb{N}$, $a \in \mathcal{M}_{p+2}$ and $c \in \mathcal{M}_p$ with $c \preccurlyeq a$. Then there exist exactly two $(p+1)$–cells $b', b'' \in \mathcal{M}_{p+1}$ such that

$$c \preccurlyeq b', \quad c \preccurlyeq b'', \qquad b' \preccurlyeq a, \quad b'' \preccurlyeq a.$$

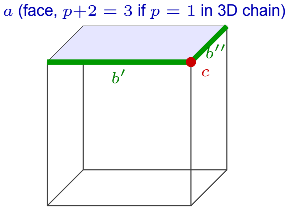
$$\begin{tikzcd}
 & a\ (p{+}2) \arrow[dl, no head] \arrow[dr, no head] & \\
b' \arrow[dr, no head] & & b'' \arrow[dl, no head] \\
 & c\ (p) &
\end{tikzcd}$$

Example $p = 0$



Diamond Property: $p = 1$ Example

Cube Slice



Interpretation

For $p = 1$ in 3D, let a be a face, c a vertex on a. Exactly two edges b', b'' on a meet at c, forming the diamond.

Relative Orientations

Theorem (Relative Orientation)

Let M be a combinatorial mesh (CM). Then there exist signs $\epsilon(a,b) \in \{-1,1\}$, for any incident pair $(a,b) \in M_{p+1} \times M_p$ with $b \prec a$ ("b is a face of a"), called the *relative orientation*, such that for each diamond

$$c \in M_p, \quad b', b'' \in M_{p+1}, \quad a \in M_{p+2}, \qquad c \preccurlyeq b', b'' \preccurlyeq a,$$

the following holds:

$$\epsilon(a,b')\,\epsilon(b',c) \;=\; -\,\epsilon(a,b'')\,\epsilon(b'',c).$$

Moreover, if an edge E has endpoints (nodes) N', N'', then

$$\epsilon(E,N') \;=\; -\,\epsilon(E,N'').$$

Example

CMC Diamond Structure (Local Incidence)

$$
\begin{tikzcd}
& \mathbf{a} \in M_{p+2} \arrow[dl, dash, "\epsilon(\mathbf{a}, \mathbf{b}')"'] \arrow[dr, dash, "\epsilon(\mathbf{a}, \mathbf{b}'')"] & \\
\mathbf{b}' \in M_{p+1} \arrow[dr, dash, "\epsilon(\mathbf{b}', \mathbf{c})"'] & & \mathbf{b}'' \in M_{p+1} \arrow[dl, dash, "\epsilon(\mathbf{b}'', \mathbf{c})"] \\
& \mathbf{c} \in M_p &
\end{tikzcd}
$$

Chains and the Boundary Operator

Definition

Let M be a combinatorial D–dimensional mesh. For $p \in \{0, 1, \dots, D\}$ let

$$C_p M := \mathsf{Free}_{\mathbb{R}}(M_p) = \left\{ \sum_{i=1}^{n} \lambda_i a_i \;\middle|\; \lambda_i \in \mathbb{R},\, a_i \in M_p \right\}.$$

The space of all chains is $C_\bullet M = \bigoplus_{p=0}^{D} C_p M$.
Given a choice of relative orientations ϵ, define

$$\partial_p : C_p M \longrightarrow C_{p-1} M, \qquad \partial_p a = \sum_{\substack{b \prec a \\ b \in M_{p-1}}} \epsilon(a, b)\, b, \quad a \in M_p.$$

If we fix the standard bases (cells ordered, with signs by ϵ), the matrix of ∂_p is denoted $\overline{\partial}_p$.

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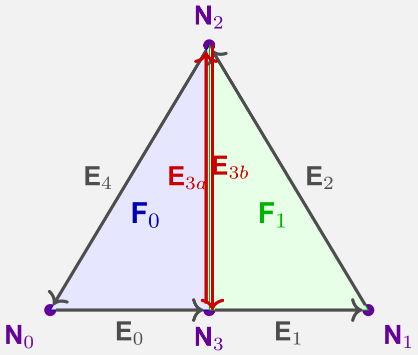
Example Mesh

Triangular with a Diagonal



Adjacent Faces F_0, F_1 Sharing Edges E_{3a}, E_{3b}

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Example

Chosen Orientations

Edges:

$$E_0 : N_0 \to N_3,$$
$$E_1 : N_3 \to N_1,$$
$$E_2 : N_1 \to N_2,$$
$$E_{3a} : N_3 \to N_2,$$
$$E_{3b} : N_2 \to N_3,$$
$$E_4 : N_2 \to N_0.$$

Faces (counterclockwise):

$$F_0 : (N_0 \to N_3 \to N_2 \to N_0),$$
$$F_1 : (N_3 \to N_1 \to N_2 \to N_3).$$

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Boundary on Edges and Faces

Edge Boundaries ($\partial_1 : C_1 \to C_0$)

$$\partial_1(E_0) = N_3 - N_0, \quad \partial_1(E_1) = N_1 - N_3, \quad \partial_1(E_2) = N_2 - N_1,$$
$$\partial_1(E_{3a}) = N_2 - N_3, \quad \partial_1(E_{3b}) = N_3 - N_2, \quad \partial_1(E_4) = N_0 - N_2.$$

Face Boundaries ($\partial_2 : C_2 \to C_1$)

$$\partial_2(F_0) = E_0 + E_{3a} + E_4,$$
$$\partial_2(F_1) = E_1 + E_2 + E_{3b}.$$

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Boundary on Edges and Faces

Matrices in the Ordered Bases

Let rows/cols be ordered as indicated under each matrix.

$$\bar{\partial}_1 = (\partial_1)^N_E = \begin{array}{c} \\ N_0 \\ N_1 \\ N_2 \\ N_3 \end{array} \left[\begin{array}{cccccc} E_0 & E_1 & E_2 & E_{3a} & E_{3b} & E_4 \\ \hline -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & -1 & 1 & 0 \end{array} \right],$$

$$\bar{\partial}_2 = (\partial_2)^E_F = \begin{array}{c} \\ E_0 \\ E_1 \\ E_2 \\ E_{3a} \\ E_{3b} \\ E_4 \end{array} \left[\begin{array}{cc} F_0 & F_1 \\ \hline 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{array} \right].$$

Nilpotency of the Boundary: $\partial \circ \partial = 0$

Theorem (Chain Complex Property)

For any combinatorial mesh M and any $p \in \mathbb{N}$,

$$\partial_p \circ \partial_{p+1} = 0 \; : \; C_{p+1}M \longrightarrow C_{p-1}M,$$

equivalently, in matrix form,

$$\overline{\partial}_p \, \overline{\partial}_{p+1} = \overline{0} \in \mathbb{R}^{|M_{p-1}| \times |M_{p+1}|}.$$

Verification in the Example

Using the previous matrices,

$$\bar{\partial}_1 \bar{\partial}_2 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Illustrating $\partial_2(\mathbf{F}) = 0$ (The boundary of a face is a null cycle)

$$\mathbf{F}_0 : \quad \partial_2(\mathbf{F}_0) = \partial_1(\mathbf{E}_0) + \partial_1(\mathbf{E}_1) + \partial_1(\mathbf{E}_2) = \mathbf{0}$$
$$\mathbf{F}_1 : \quad \partial_2(\mathbf{F}_1) = -\partial_1(\mathbf{E}_0) - \partial_1(\mathbf{E}_1) + \partial_1(\mathbf{E}_3) = \mathbf{0}$$

where $\mathbf{F}_i$ are the 2-cells (faces) and $\mathbf{E}_i$ are the 1-cells (edges).
∂_k is the boundary operator from k-cells to $(k-1)$-cells.

Remark

Boundary Operator and Chain Complex

For a combinatorial D–mesh M, the chain spaces $(C_pM)_{p=0}^{D}$ with boundary maps

$$C_DM \xrightarrow{\partial_D} C_{D-1}M \xrightarrow{\partial_{D-1}} \cdots \xrightarrow{\partial_2} C_1M \xrightarrow{\partial_1} C_0M$$

form a chain complex.

Chain complex property: $\partial_p \circ \partial_{p+1} = 0, \quad \forall p \in \{0, \ldots, D-1\}.$

$$C_DM \xrightarrow{\partial_D} C_{D-1}M \xrightarrow{\partial_{D-1}} \cdots \xrightarrow{\partial_2} C_1M \xrightarrow{\partial_1} C_0M$$

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Cochains and Coboundary

Definition (Cochains and Coboundary)

Let (M, ϵ) be a D–dimensional CM with a fixed system of relative orientations. For $p \in \{0, \dots, D\}$, the space of p–cochains is the $\mathbb{R}$–linear dual

$$C^p M := (C_p M)^*, \qquad C^\bullet M := \bigoplus_{p=0}^{D} C^p M.$$

Choose dual bases concretely:

$$\begin{aligned} p = 0 :&\ (N^0, N^1, \dots) \text{ dual to } (N_0, N_1, \dots), \\ p = 1 :&\ (E^0, E^1, \dots) \text{ dual to } (E_0, E_1, \dots), \\ p = 2 :&\ (F^0, F^1, \dots) \text{ dual to } (F_0, F_1, \dots), \\ p = 3 :&\ (V^0, V^1, \dots) \text{ dual to } (V_0, V_1, \dots). \end{aligned}$$

Cochains and Coboundary

Definition (Cochains and Coboundary)

The coboundary operators $\delta_p : C^p M \to C^{p+1} M$ are the duals of ∂_{p+1}:

$$\delta_p \sigma \ = \ \sigma \circ \partial_{p+1} \in C^{p+1} M, \qquad \bar{\delta}_p \ = \ \bar{\partial}_{p+1}^{\mathsf{T}}.$$

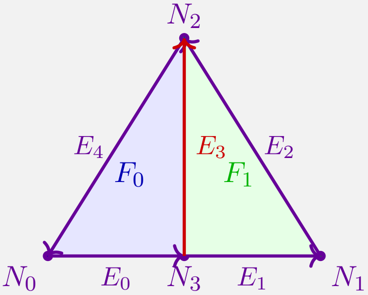
Cochain Complex Property

$$\delta_{p+1} \circ \delta_p \ = \ 0, \qquad \forall\, p \in \{0, \ldots, D-1\}.$$

$$C^0 M \xrightarrow{\delta_0} C^1 M \xrightarrow{\delta_1} \cdots \xrightarrow{\delta_{D-2}} C^{D-1} M \xrightarrow{\delta_{D-1}} C^D M$$

Coboundary Matrices and Actions

Triangular Mesh with a Diagonal



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Example

Coboundary Matrices $\bar{\delta}_0, \bar{\delta}_1$

From the consistent boundary operators $\bar{\partial}_1 \in \mathbb{R}^{4\times 5}$, $\bar{\partial}_2 \in \mathbb{R}^{5\times 2}$:

$$\bar{\delta}_0 = \bar{\partial}_1^{\mathsf{T}} = \begin{array}{c} \\ E^0 \\ E^1 \\ E^2 \\ E^3 \\ E^4 \end{array} \left[\begin{array}{cccc} N^0 & N^1 & N^2 & N^3 \\ \hline -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & -1 & 0 \end{array}\right],$$

$$\bar{\delta}_1 = \bar{\partial}_2^{\mathsf{T}} = \begin{array}{c} \\ F^0 \\ F^1 \end{array} \left[\begin{array}{ccccc} E^0 & E^1 & E^2 & E^3 & E^4 \\ \hline 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 & 0 \end{array}\right].$$

Example

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Actions on Cochains

Let $\sigma \in C^0 M$ and $\rho \in C^1 M$. Then

$$(\delta_0\sigma)(E_3) = \sigma\big(\partial_1 E_3\big) = \sigma(N_2 - N_3) = \sigma(N_2) - \sigma(N_3),$$
$$(\delta_1\rho)(F_0) = \rho\big(\partial_2 F_0\big) = \rho(E_0 + E_3 + E_4) = \rho(E_0) + \rho(E_3) + \rho(E_4),$$
$$(\delta_1\rho)(F_1) = \rho\big(\partial_2 F_1\big) = \rho(E_1 + E_2 - E_3) = \rho(E_1) + \rho(E_2) - \rho(E_3).$$

Historical Note: Georges de Rham

Georges de Rham (1903–1990) was a Swiss mathematician who pioneered the connection between differential forms and topology. His work led to the **de Rham theorem**, establishing the equivalence between differential and topological cohomology - the foundation of the modern de Rham map. see next

de Rham Map

Definition

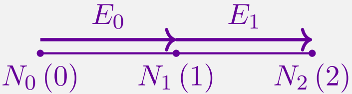
Let X be a smooth manifold with dim $X = D$, and let M be a CM embedded by $\varphi : M \hookrightarrow \mathcal{P}(X)$. For $p \in \{0, \ldots, D\}$, define

$$R_p : \Omega^p(X) \to C^p M, \qquad (R_p\omega)(c) := \int_{\varphi(c)} \omega, \quad c \in M_p,$$

and extend by linearity to chains.

Example on the Interval $X = [0, 2]$

Nodes: N_0, N_1, N_2 at $x = 0, 1, 2$. Edges: $E_0 = [0, 1]$, $E_1 = [1, 2]$ with the natural left-to-right orientation.



Example

Concrete Evaluation

For $p = 0$ (functions $f \in \Omega^0(X)$):

$$(R_0 f)(N_i) = \int_{\{x_i\}} f \;=\; f(x_i), \qquad x_0 = 0,\ x_1 = 1,\ x_2 = 2.$$

For $p = 1$ ($\omega = g(x)\,dx \in \Omega^1(X)$):

$$(R_1\omega)(E_0) = \int_0^1 g(x)\,dx, \qquad (R_1\omega)(E_1) = \int_1^2 g(x)\,dx.$$

Thus $R_0 : \Omega^0 \to C^0 M$ samples at nodes; $R_1 : \Omega^1 \to C^1 M$ integrates along oriented edges.

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Discrete Stokes–Cartan

Theorem (de Rham–Coboundary Commutativity)

For $p \in \{0, \ldots, D-1\}$,

$$R_{p+1} \circ d_p = \delta_p \circ R_p.$$

$$\begin{tikzcd} \Omega^p(X) \arrow[r, "d_p"] \arrow[d, "R_p"'] & \Omega^{p+1}(X) \arrow[d, "R_{p+1}"] \\ C^p M \arrow[r, "\delta_p"'] & C^{p+1} M \end{tikzcd}$$

Specialization (e.g. $D = 3$)

For $p = 0, 1, 2$ the same commutative diagram holds with $\Omega^p(\mathbb{R}^3)$ and $C^p M$ (vertices, edges, faces) embedded in $X = \mathbb{R}^3$ by φ.

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Discrete and Smooth Trace

Definition (Discrete Trace)

Let $L \subseteq M$ be a submesh. The trace

$$\mathrm{tr}_p : C^p M \longrightarrow C^p L, \qquad (\mathrm{tr}_p\, \sigma)(c) \;=\; \sigma(c), \quad c \in L_p,$$

is restriction of cochains to L.

Compatibility (Smooth vs Discrete Trace)

Let $\mathrm{tr}_p^{\mathrm{sm}} : \Omega^p(X) \to \Omega^p(\partial X)$ be the smooth trace (pullback to the boundary). Then the de Rham maps commute with trace:

$$\mathrm{tr}_p \circ R_p \;=\; R_p^L \circ \mathrm{tr}_p^{\mathrm{sm}},$$

where $R_p^L : \Omega^p(\partial X) \to C^p L$ is the de Rham map on the submesh L.

Commutative Diagram

$$\begin{tikzcd}
\Omega^p(X) \arrow[r, "\mathrm{tr}_p^{\mathrm{sm}}"] \arrow[d, "R_p"'] & \Omega^p(\partial X) \arrow[d, "R_p^L"] \\
C^p M \arrow[r, "\mathrm{tr}_p"'] & C^p L
\end{tikzcd}$$

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Summary of Combinatorial Foundations

Discrete Geometry Essentials

- p**–cells:** Basic building blocks — single points ($p=0$) or open p–balls ($p>0$).
- **Regular cell complex:** A collection of cells closed under intersection and face closure, i.e. cl$(a) = \bigcup_{b\subseteq a} b$.
- **Combinatorial mesh:** A partially ordered set $(M, \preccurlyeq)$ that can be embedded into a smooth manifold X such that its image forms a regular cell complex.
- **Polytopes:** Meshes with one top D–cell — segments (1D), polygons (2D), cubes/tetrahedra (3D), etc.
- **Diamond property:** For $a \in M_{p+2}, c \in M_p$ with $c \prec a$, there exist exactly two $(p+1)$–cells b' and b'' such that $c \prec b', b'' \prec a$.

Summary of Chain Complex Structure

Orientation and Boundary Operators

- **Relative orientation:** $\epsilon(a,b) \in \{\pm 1\}$ defines the signed incidence between adjacent cells satisfying $\epsilon(a,b')\epsilon(b',c) = -\epsilon(a,b'')\epsilon(b'',c)$.
- **Chains:** $C_pM = \text{Free}_{\mathbb{R}}(M_p)$ - linear combinations of oriented p–cells.
- **Boundary operator:**

$$\partial_p(a) = \sum_{b \prec a} \epsilon(a,b)\, b, \qquad \partial_{p+1} \circ \partial_{p+2} = 0.$$

- **Example (triangular cell with diagonal):** Explicit ∂_1, ∂_2 matrices verified $\overline{\partial}_1 \overline{\partial}_2 = 0$.

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Summary of Discrete Differential

Dual Spaces and Operators

- **Cochains:** $C^p M = (C_p M)^*$ - linear duals of p–chains.
- **Coboundary operator:** $\delta_p = \partial_{p+1}^T$ defines the discrete analogue of the exterior derivative, satisfying $\delta_{p+1} \circ \delta_p = 0$.
- **Example:** Explicit matrices $\overline{\delta}_0, \overline{\delta}_1$ act on cochains:

$$(\delta_0 \sigma)(E_i) = \sigma(\partial_1 E_i), \quad (\delta_1 \rho)(F_j) = \rho(\partial_2 F_j).$$

- **de Rham map:** $R_p : \Omega^p(X) \to C^p M$ with

$$R_{p+1} \circ d = \delta_p \circ R_p,$$

  establishing equivalence between smooth and discrete calculus.
- **Trace consistency:** $\mathrm{tr}_p^{\text{disc}} \circ R_p^M = R_p^L \circ \mathrm{tr}_p^{\text{smooth}}$.

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Thanks