

Combinatorial Mesh Calculus (CMC): Lecture 13

.....

Lectured by: **Dr. Kiprian Berbatov**¹

Lecture Notes Compiled by: **Muhammad Azeem**¹

Under the supervision of: **Prof. Andrey P. Jivkov**¹

¹ Department of Mechanical and Aerospace Engineering, The University of Manchester, Oxford Road,
Manchester M13 9PL, UK



Definition

Let (M, g) be a compact oriented D -dimensional Riemannian manifold.

1. The **measure** of M (length, area, or volume for $D = 1, 2, 3$) is

$$\mu(M) = \int_M \text{vol}_{(M,g)} > 0.$$

2. The **Riemann integral** of a smooth function $f \in \mathcal{F}(M)$ is

$$I(f) = \int_M f \text{vol}_{(M,g)}.$$

Definition

For $D = 1$, $M = [a, b]$, this reduces to the classical integral

$$I(f) = \int_a^b f(x) dx.$$

3 The inner product of two p -forms $\omega, \eta \in \Omega^p(M)$ with $p \in \{0, 1, \dots, D\}$, is

$$\begin{aligned} \langle \omega, \eta \rangle_p &:= I(g_p(\omega, \eta)). \\ &= \int_M g_p^*(\omega, \eta) \operatorname{vol}_{(M, g)}. \end{aligned}$$

Example: Area on the Sphere S_r^2

Setup

Let $D = 2$, $M = S_r^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = r^2\}$ with the induced orientation and metric from $\mathbb{R}^3$. In spherical coordinates:

$$\begin{aligned}x &= r \sin \theta \cos \phi, & y &= r \sin \theta \sin \phi, & z &= r \cos \theta, \\ \theta &\in (0, \pi), & \phi &\in (0, 2\pi).\end{aligned}$$

The induced metric is $g = r^2(d\theta^2 + \sin^2 \theta d\phi^2)$, and the volume form is

$$\text{vol}_{(M,g)} = r^2 \sin \theta d\theta \wedge d\phi.$$

Example

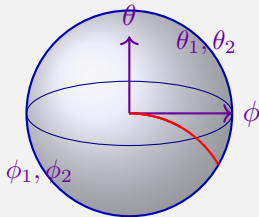
Computation of Area of a Spherical Quadrilateral

For $\theta \in [\theta_1, \theta_2]$ and $\phi \in [\phi_1, \phi_2]$,

$$\begin{aligned} A &= \int_{[\theta_1, \theta_2] \times [\phi_1, \phi_2]} r^2 \sin \theta \, d\theta \wedge d\phi \\ &= r^2 \int_{\phi_1}^{\phi_2} \int_{\theta_1}^{\theta_2} \sin \theta \, d\theta \, d\phi \\ &= r^2 (\phi_2 - \phi_1) (\cos \theta_1 - \cos \theta_2). \end{aligned}$$

This agrees with the Frobenius theorem, ensuring the integration of a 2-form depends only on the orientation of M .

Visualization



Codifferential Operator Manifolds

Definition

Let (M, g) be a compact oriented D -dimensional Riemannian manifold, and $p \in \{1, 2, \dots, D\}$. Denote by

$\overset{\circ}{\Omega}^p(M) = \{\omega \in \Omega^p(M) \mid \text{tr}_{\partial M} \omega = 0\}$ the space of p -forms vanishing on the boundary.

The **codifferential**

$$d_p^* : \overset{\circ}{\Omega}^p(M) \longrightarrow \overset{\circ}{\Omega}^{p-1}(M)$$

is the adjoint of d_{p-1} restricted to $\overset{\circ}{\Omega}^{p-1}(M)$:

$$\langle d_p^* \omega, \eta \rangle_{p-1} = \langle \omega, d_{p-1} \eta \rangle_p, \quad \forall \omega \in \overset{\circ}{\Omega}^p(M), \eta \in \overset{\circ}{\Omega}^{p-1}(M).$$

Expression via Hodge Star

Equivalent Expression

For any $p \in \{1, \dots, D\}$,

$$d_{D-p}^* \circ \star_p = (-1)^{p+1} \star_{p+1} \circ d_p,$$

or equivalently:

$$\begin{aligned} d_{D-p}^* &= (-1)^{p+1} \star_{p+1} \circ d_p \circ \star_p^{-1} \\ &= (-1)^{p+1+p(D-p)} \star_{p+1} \circ d_p \circ \star_{D-p}. \end{aligned}$$

$$\begin{array}{ccc} \Omega^p(M) & \xrightarrow{d_p} & \Omega^{p+1}(M) \\ \star_p \downarrow & & \downarrow \star_{p+1} \\ \Omega^{D-p}(M) & \xrightarrow{(-1)^{p+1} d_{D-p}^*} & \Omega^{D-p-1}(M) \end{array}$$

Codifferential: General Identity

Setup and Symbols

(M, g) : oriented D -dimensional Riemannian manifold.

$\Omega^p(M)$: p -forms on M ,

$d_p : \Omega^p(M) \rightarrow \Omega^{p+1}(M)$.

$\star_p : \Omega^p(M) \rightarrow \Omega^{D-p}(M)$ (Hodge star),

$\star_p^{-1} = (-1)^{p(D-p)} \star_{D-p}$.

Inner product: $g_p^*(\omega, \eta) \text{vol}_g = \omega \wedge \star_p \eta$,

$\langle \omega, \eta \rangle_p = \int_M g_p^*(\omega, \eta) \text{vol}_g$.

Codifferential

For $p \in \{0, 1, \dots, D-1\}$, the adjoint

$$d_{D-p}^* = (-1)^{p+1+p(D-p)} \star_{p+1} \circ d_p \circ \star_{D-p},$$

$$d_{D-p}^* : \Omega^{D-p}(M) \longrightarrow \Omega^{D-p-1}(M).$$

$f \in \Omega^0(M) = \mathcal{F}(M)$ (smooth scalar function).

$\omega \in \Omega^1(M)$ (one-form; via g corresponds to a vector field by $\omega = X^\flat$).

$\eta \in \Omega^2(M)$ (two-form; in $D = 3$, $\eta = \star_1 Y^\flat$
encodes an oriented flux field).

Specialization to $D = 3$: Explicit d and d^*

Exterior derivatives in $D = 3$

$$d_0 : \Omega^0(M) \rightarrow \Omega^1(M),$$

$$d_0(f) = df,$$

$$d_1 : \Omega^1(M) \rightarrow \Omega^2(M),$$

$$d_1(\omega) = d\omega,$$

$$d_2 : \Omega^2(M) \rightarrow \Omega^3(M),$$

$$d_2(\eta) = d\eta,$$

$$d_3 : \Omega^3(M) \rightarrow \Omega^4(M) = 0,$$

$$d_3 = 0.$$

Adjoint from the general identity

Using $d_{D-p}^* = (-1)^{p+1+p(D-p)} \star_{p+1} d_p \star_{D-p}$ with $D = 3$:

$$\begin{aligned} \text{for } p = 2 : \quad d_1^* &= (-1)^{2+1+2(1)} \star_3 d_2 \star_1 \\ &= (-1)^5 \star_3 d_2 \star_1 \\ &= - \star_3 d_2 \star_1, \end{aligned}$$

Specialization to $D = 3$: Explicit d and d^*

Adjoint from the general identity

$d_1^* : \Omega^1(M) \rightarrow \Omega^0(M)$ (“negative divergence” under the standard identifications).

$$\begin{aligned} \text{for } p = 1 : \quad d_2^* &= (-1)^{1+1+1(2)} \star_2 d_1 \star_2 \\ &= (-1)^4 \star_2 d_1 \star_2 \\ &= \star_2 d_1 \star_2 \end{aligned}$$

$d_2^* : \Omega^2(M) \rightarrow \Omega^1(M)$ (“curl” under the standard identifications).

$$\begin{aligned} \text{for } p = 0 : \quad d_3^* &= (-1)^{0+1+0} \star_1 d_0 \star_3 \\ &= - \star_1 d_0 \star_3, \end{aligned}$$

$d_3^* : \Omega^3(M) \rightarrow \Omega^2(M)$ (co-gradient of a density 3-form).

Interpretation in $\mathbb{R}^3$ and Schematic

Vector-calculus dictionary (with this sign convention)

$$d_0 \rightsquigarrow \text{gradient } (\nabla f),$$

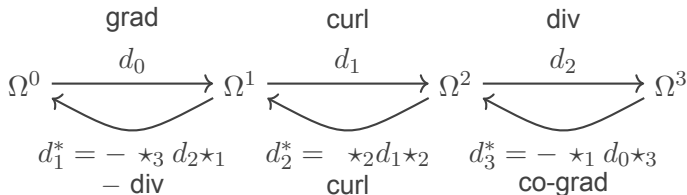
$$d_1 \rightsquigarrow \text{curl } (\nabla \times A),$$

$$d_2 \rightsquigarrow \text{divergence } (\nabla \cdot F),$$

$$d_1^* = - \star_3 d_2 \star_1 \rightsquigarrow - \text{div}(A),$$

$$d_2^* = \star_2 d_1 \star_2 \rightsquigarrow \nabla \times F,$$

$$d_3^* = - \star_1 d_0 \star_3 .$$



Domains/Codomains

$$D = 3$$

$$f \in \Omega^0 \xrightarrow{d_0} df \in \Omega^1,$$

$$\omega \in \Omega^1 \xrightarrow{d_1} d\omega \in \Omega^2,$$

$$\eta \in \Omega^2 \xrightarrow{d_2} d\eta \in \Omega^3,$$

$$\alpha \in \Omega^1 \xrightarrow{d_1^*} d_1^* \alpha \in \Omega^0,$$

$$\beta \in \Omega^2 \xrightarrow{d_2^*} d_2^* \beta \in \Omega^1,$$

$$\gamma \in \Omega^3 \xrightarrow{d_3^*} d_3^* \gamma \in \Omega^2.$$

Setup and Field Quantities

Geometric–Physical Setting

Let $D \in \mathbb{N}$. Let (M, g) be a compact, oriented D –dimensional Riemannian manifold with nonempty boundary ∂M (spatial domain). Let $t_0 \in \mathbb{R}$ and $I = [t_0, \infty)$ (time interval). We study the transport of an extensive quantity (mass, charge, energy, ...) in M over I .

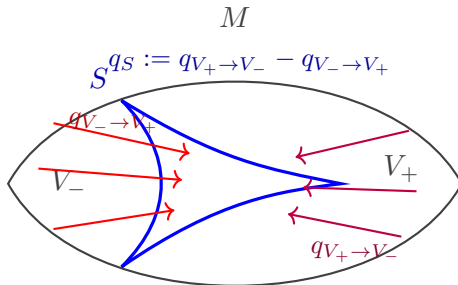
Differential–Form Fields (time–dependent)

Amount (density D –form): $Q \in C^\infty(I, \Omega^D M)$ so that $\int_V Q(t)$ is the amount in $V \subseteq M$.

Flow rate (flux $(D-1)$ –form): $q \in C^\infty(I, \Omega^{D-1} M)$, so that $\int_S q(t)$ is net flow through S^{D-1} .

Internal production (source D –form): $f \in C^\infty(I, \Omega^D M)$, so that $\int_V f(t)$ is total production in V .

Flux Across an Internal Interface



Time-Integrated Net Flux Through S

For $[t_1, t_2] \subset I$,

Net flux on $[t_1, t_2]$:

$$\int_{t_1}^{t_2} \left(\int_S q(t) \right) dt.$$

Continuity Law: Integral and Differential Forms

Integral Balance on Any Subregion $V \subseteq M$

For $[t_1, t_2] \subset I$,

$$\underbrace{\int_V Q(t_2) - \int_V Q(t_1)}_{\text{amount change in } V} = \underbrace{\int_{t_1}^{t_2} \left(\int_V f(t) \right) dt}_{\text{internal production}} - \underbrace{\int_{t_1}^{t_2} \left(\int_{\partial V} q(t) \right) dt}_{\text{total outflow through } \partial V}.$$

Using Stokes–Cartan on ∂V ,

$$\int_{\partial V} q(t) = \int_V d_{D-1} q(t).$$

Continuity Law

Integral Balance on Any Subregion $V \subseteq M$

Assuming smoothness, differentiate under the integral in time:

$$\int_V \frac{\partial Q}{\partial t}(t) = \int_V (f(t) - d_{D-1}q(t)).$$

Localization (“Dropping the Integrals”)

Because the equality holds for *all* subregions V and for all t , the integrands must agree pointwise:

$$\boxed{\frac{\partial Q}{\partial t} = f - d_{D-1}q} \quad \text{on } M \times I.$$

This is the differential–form version of the continuity law.

Boundary Sign Convention and Stokes on Moving Slices

Orientation and Signs

With the outward orientation on ∂V ,

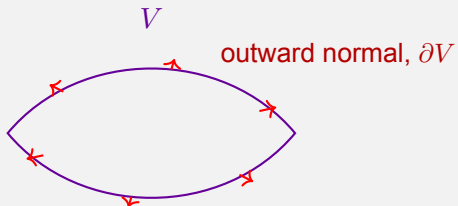
$$\int_{\partial V} q = \int_V d_{D-1}q \quad \Rightarrow \quad \text{outflow (through } \partial V) = \int_V d_{D-1}q.$$

Thus,

$$\frac{\partial Q}{\partial t} = f - d_{D-1}q$$

encodes “time rate of change = production – outflow”.

Visualization



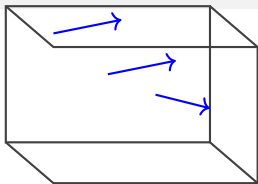
Correspondence with Vector Calculus

A Common Identification

In $D = 3$, write $Q = \tilde{\rho} dV$ (amount density $\tilde{\rho}$ times 3-volume form), and represent a vector flux field $\tilde{F}$ via the flux 2-form $q = \iota_{\tilde{F}} \text{vol}$. Then

$$\frac{\partial Q}{\partial t} = f - d_2 q \iff \frac{\partial \tilde{\rho}}{\partial t} dV = \tilde{f} dV - (\nabla \cdot \tilde{F}) dV$$

$$\iff \boxed{\frac{\partial \tilde{\rho}}{\partial t} = \tilde{f} - \nabla \cdot \tilde{F}}.$$



flux field $\tilde{F}$, $q = \iota_{\tilde{F}} \text{vol}$

Neumann Boundary and Prescribed Flux

Boundary Portion and Data

Let $\Gamma_N \subseteq \partial M$ be a $(D-1)$ -dimensional smooth submanifold (Neumann boundary). A *prescribed flux* (i.e. *given as boundary input*) is a $(D-1)$ -form

$$g_N \in C^\infty(I, \Omega^{D-1} \Gamma_N) \quad \text{with physical unit } [X T^{-1}],$$

interpreted so that $\int_S g_N(t)$ equals the net amount per unit time crossing $S \subset \Gamma_N$ at time t .

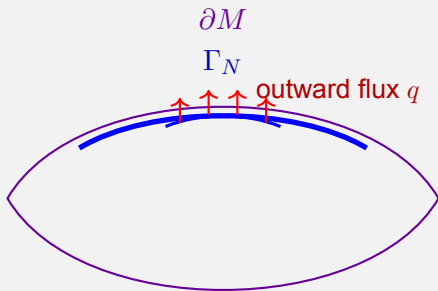
Neumann Boundary Condition (NBC)

The total flux $q \in C^\infty(I, \Omega^{D-1} M)$ satisfies

$$\text{tr}_{\Gamma_N} q = g_N \quad \text{on } \Gamma_N \times I.$$

Neumann Boundary and Prescribed Flux

Boundary Portion and Data



Potential, Diffusion, and Advection

Potential and Total Flux Split

Let $u \in C^\infty(I, \Omega^0 M)$ be the *potential* (units $[Y]$), e.g. temperature, concentration, pressure, or electric potential. Split the total flux as

$$q = q_D + q_A,$$

where q_D is the *diffusive* part and q_A is the *advective* part.

Diffusive (constitutive) law — isotropic & anisotropic

With $du \in \Omega^1 M$ and Hodge star $\star : \Omega^1 \rightarrow \Omega^{D-1}$,

$$\text{(isotropic)} \quad q_D = -\kappa \star (du), \quad \kappa \in C^\infty(M), \kappa > 0;$$

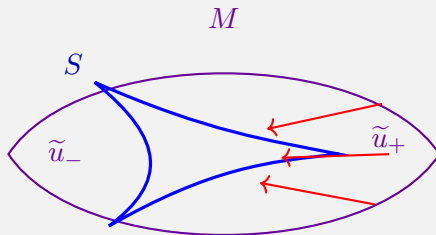
$$\text{(anisotropic)} \quad q_D = -\star(\tilde{\kappa}(du)), \quad \tilde{\kappa} : \Omega^1 M \rightarrow \Omega^1 M \\ \text{(SPD (1,1)-tensor).}$$

Potential, Diffusion, and Advection

Diffusive (constitutive) law — isotropic & anisotropic

Equivalently, define $\kappa : \Omega^{D-1}M \rightarrow \Omega^{D-1}M$ by the identity

$$\boxed{\star_{D-1} \circ \tilde{\kappa} = \kappa \circ \star_1} \quad \Rightarrow \quad q_D = -\kappa \star (du).$$



$$q|_S \sim \tilde{u}_+ - \tilde{u}_-$$

Advective Flux via Volume Flux Form

Let $\tilde{v}$ be a velocity vector field; its *volume flux form* is

$$v := \iota_{\tilde{v}} \text{vol} \in \Omega^{D-1}M \quad (\text{units } [L^D T^{-1}]).$$

If $Q \in \Omega^D M$ is the amount D -form, then

$$q_A = (\star Q) v \quad (\text{scalar density } \star Q \times \text{volume flux form } v).$$

Volumetric Capacity

Relate amount and potential by a capacity coefficient:

$$\tilde{\pi} : \Omega^0 M \rightarrow \Omega^0 M, \quad Q = \star(\tilde{\pi} u).$$

Equivalently, $\exists \pi : \Omega^D M \rightarrow \Omega^D M$ with $\boxed{\pi \circ \star_0 = \star_0 \circ \tilde{\pi}},$

$$\Rightarrow Q = \pi(\star u).$$

Hence

$$\frac{\partial Q}{\partial t} = \pi\left(\star \frac{\partial u}{\partial t}\right) = \star\left(\tilde{\pi} \frac{\partial u}{\partial t}\right).$$

Dirichlet vs. Neumann

Let $\Gamma_D, \Gamma_N \subseteq \partial M$ be relatively open, disjoint, with

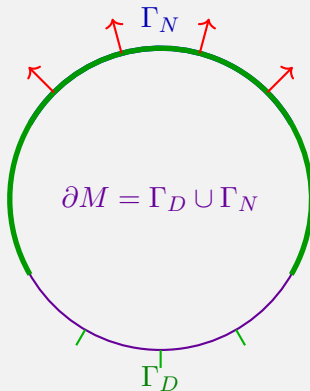
$$\Gamma_D \cup \Gamma_N = \partial M, \quad \dim(\Gamma_D \cap \Gamma_N) \leq D - 2.$$

Dirichlet boundary condition (DBC):

$$\mathrm{tr}_{\Gamma_D} u = u_D, \quad u_D \in C^\infty(I, \Omega^0 \Gamma_D).$$

Neumann boundary condition (NBC):

$$\mathrm{tr}_{\Gamma_N} q = g_N, \quad g_N \in C^\infty(I, \Omega^{D-1} \Gamma_N).$$



Governing Equations (Summary)

Bulk Law (Continuity)

$$\frac{\partial Q}{\partial t} = f - d_{D-1}q \quad \text{in } M \times I.$$

Constitutive Relations

Capacity: $Q = \star(\tilde{\pi} u) = \pi(\star u), \quad \pi \circ \star_0 = \star_0 \circ \tilde{\pi}.$

Diffusion: $q_D = -\star(\tilde{\kappa}(du)) = -\kappa \star(du), \quad \star_{D-1} \circ \tilde{\kappa} = \kappa \circ \star_1.$

Advection: $q_A = (\star Q)v, \quad v = \iota_{\tilde{v}} \text{vol} \in \Omega^{D-1}M.$

Governing Equations (Summary)

Boundary Conditions and Unknowns

Unknowns: $u \in C^\infty(I, \Omega^0 M)$, $Q \in C^\infty(I, \Omega^D M)$,
 $q \in C^\infty(I, \Omega^{D-1} M)$.

Dirichlet on Γ_D : $\text{tr}_{\Gamma_D} u = u_D$.

Neumann on Γ_N : $\text{tr}_{\Gamma_N} q = g_N$.

Total flux: $q = q_D + q_A$.

Primal Strong Model

Continuity and Constitutive Content (recall)

Continuity:
$$\frac{\partial Q}{\partial t} = f - d_{D-1}q,$$

Amount–potential link (capacity):
$$Q = \star(\tilde{\pi} u)$$

(equivalently $\tilde{\pi} u = \star_D Q$),

Flux split:
$$q = q_D + q_A,$$

Diffusion (anisotropic):
$$q_D = - \star(\tilde{\kappa} du)$$

(isotropic: $q_D = - \kappa \star du$),

Advection:
$$q_A = (\star Q) v, \quad v = \iota_{\tilde{v}} \text{vol} \in \Omega^{D-1}M.$$

Primal Strong Model

Substitute Q, q in terms of u

$$\begin{aligned}\frac{\partial}{\partial t} \left(\star(\tilde{\pi} u) \right) &= f - d_{D-1} \left(- \star(\tilde{\kappa} du) + (\star Q) v \right) \\ &= f + d_{D-1} \star(\tilde{\kappa} du) - d_{D-1}((\star Q) v) \\ &= f + d_{D-1} \star(\tilde{\kappa} du) - d_{D-1} \left((\star \star(\tilde{\pi} u)) v \right),\end{aligned}$$

where in the last line we used $Q = \star(\tilde{\pi} u)$.

Primal strong equation for u

In isotropic diffusion ($\tilde{\kappa} = \kappa \text{ id}$), the second term is $d_{D-1}(\kappa \star du)$.

Equivalent Form Using $\tilde{\pi}u = \star_D Q$

Express everything through Q and then back to u

$$\tilde{\pi}u = \star_D Q \iff \star(\tilde{\pi}u) = \star \star_D Q = \sigma_D Q,$$

where $\sigma_D = \pm 1$ depends on the metric signature and D (in Euclidean D -RM, $\sigma_D = +1$). Hence

$$\frac{\partial}{\partial t} \left(\star(\tilde{\pi}u) \right) = \sigma_D \frac{\partial Q}{\partial t}.$$

Using the continuity equation $\frac{\partial Q}{\partial t} = f - d_{D-1}q$ and $q = -\star(\tilde{\kappa}du) + (\star Q)v$, we get

$$\begin{aligned} \sigma_D \frac{\partial Q}{\partial t} &= f - d_{D-1} \left(-\star(\tilde{\kappa}du) + (\star Q)v \right) \\ &= f + d_{D-1} \star(\tilde{\kappa}du) - d_{D-1}((\star Q)v). \end{aligned}$$

Replacing Q by $\star(\tilde{\pi}u)$ recovers the primal equation in u from the previous slide.

Steady assumptions and model

$$0 = f - d_{D-1}q \quad \Rightarrow \quad f = d_{D-1}q.$$
$$f = d_{D-1}(-\kappa \star du) = -\kappa d_{D-1}(\star du).$$

Manufactured Steady Problem

Choose $M = [0, 1]^3 \subset \mathbb{R}^3$, standard Euclidean metric

Let $u(x, y, z) = x^2 + y^2 + z^2$ and $\kappa = 2$ (constant).

$$du = 2x dx + 2y dy + 2z dz,$$

$$\star dx = dy \wedge dz, \quad \star dy = dz \wedge dx, \quad \star dz = dx \wedge dy,$$

$$\star du = 2x dy \wedge dz + 2y dz \wedge dx + 2z dx \wedge dy,$$

$$\begin{aligned} q &= -\kappa \star du = -2 \left(2x dy \wedge dz + 2y dz \wedge dx + 2z dx \wedge dy \right) \\ &= -4x dy \wedge dz - 4y dz \wedge dx - 4z dx \wedge dy. \end{aligned}$$

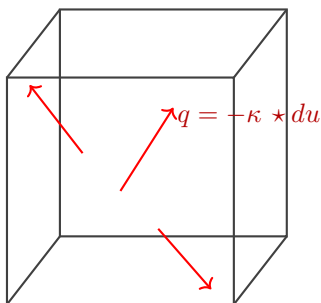
Finally,

$$\begin{aligned} f = dq &= -\kappa d(\star du) = -2 (\Delta u) dx \wedge dy \wedge dz = -2 \cdot 6 dx \wedge dy \wedge dz \\ &= -12 dx \wedge dy \wedge dz. \end{aligned}$$

Thus the manufactured source is the constant 3-form

$f = -12 dx \wedge dy \wedge dz$ on M .

Flux Sketch (steady diffusion in a cube)



$$M = [0, 1]^3$$

Main Concepts

- $\mu(M) = \int_M \text{vol}_{(M,g)}$ gives measure (length/area/volume) of (M, g) .
- $\langle \omega, \eta \rangle_p = \int_M g_p^*(\omega, \eta) \text{vol}_{(M,g)}$ defines inner product on $\Omega^p(M)$.
- Codifferential d^* is the adjoint of d :

$$\langle d_p^* \omega, \eta \rangle_{p-1} = \langle \omega, d_{p-1} \eta \rangle_p.$$

- Relationship with Hodge star:

$$d_{D-p}^* \circ \star_p = (-1)^{p+1} \star_{p+1} \circ d_p.$$

- For $D = 3$: (d_0, d_1, d_2) correspond to $(\nabla, \nabla \times, \nabla \cdot)$ and (d_1^*, d_2^*) to their adjoints.

Summary

Main Concepts

- Neumann data prescribes the *flux* $(D-1)$ -form on Γ_N , Dirichlet data prescribes the *potential* on Γ_D .
- Diffusion: $q_D = -\star(\tilde{\kappa} du)$ (or $-\kappa \star du$); Advection: $q_A = (\star Q) v$ with $v = \iota_{\tilde{v}} \text{vol}$.
- Capacity links amount and potential: $Q = \star(\tilde{\pi} u)$.
- Continuity in forms: $\frac{\partial Q}{\partial t} = f - d_{D-1} q$, closed by the constitutive laws and boundary conditions.

Summary

Main Concepts

- Amount $Q(t) \in \Omega^D M$, flux $q(t) \in \Omega^{D-1} M$, production $f(t) \in \Omega^D M$ model transport on (M, g) .
- Integral balance on any $V \subseteq M$ and Stokes–Cartan yield the local continuity law

$$\frac{\partial Q}{\partial t} = f - d_{D-1} q.$$

- In $D = 3$, with $Q = \tilde{\rho} dV$ and $q = \iota_{\tilde{F}} \text{vol}$, this becomes

$$\frac{\partial \tilde{\rho}}{\partial t} = \tilde{f} - \nabla \cdot \tilde{F}.$$

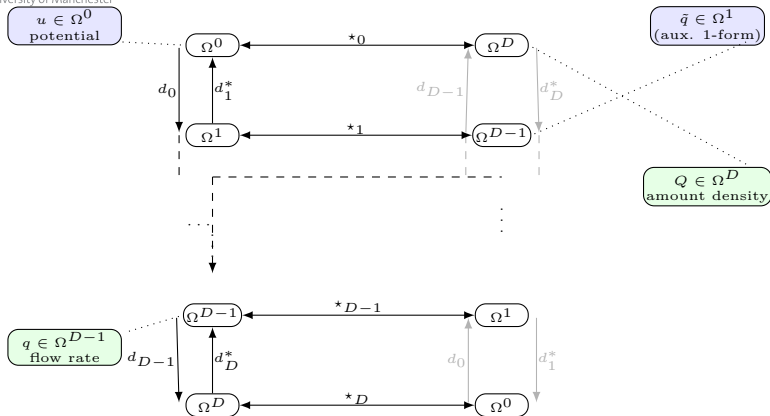
Main Concepts

- Primal strong model (exterior–calculus form):

$$\frac{\partial}{\partial t} (\star (\tilde{\pi} u)) - d_{D-1} \star (\tilde{\kappa} du) + d_{D-1} ((\star \star (\tilde{\pi} u)) v) = f.$$

- Steady, no advection: $f = dq = -\kappa d(\star du)$; in $\mathbb{R}^3$ this is $f = -\kappa \Delta u \text{ vol}$.
- Manufactured 3D example with $u = x^2 + y^2 + z^2$, $\kappa = 2$:
 $q = -4x dy \wedge dz - 4y dz \wedge dx - 4z dx \wedge dy,$
 $f = -12 dx \wedge dy \wedge dz.$

Global Picture



Codifferential:

$$d_{D-p}^* = (-1)^{p+1+p(D-p)} *_{p+1} \circ d_p \circ *_{D-p}$$

Inner product: $g_p^*(\omega, \eta) \text{ vol} = \omega \wedge *p \eta$

Hodge involution:

$$*_{D-p} \circ *p = (-1)^{p(D-p)} \text{id}_{\Omega p}$$

Thanks

