Combinatorial Mesh Calculus (CMC): Lecture 12

Lectured by: Dr. Kiprian Berbatov[1]
Lecture Notes Compiled by: Muhammad Azeem[1]
Under the supervision of: Prof. Andrey P. Jivkov[1]

[1] Department of Mechanical and Aerospace Engineering, The University of Manchester, Oxford Road, Manchester M13 9PL, UK

Area of a Disk via Green's Theorem

Setup

For $r > 0$, let

$$\begin{aligned} M &= \overline{B_r}(0,0) = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq r^2\}, \\ \partial M &= S^1_r = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = r^2\}. \end{aligned}$$

The area form on $\mathbb{R}^2$ is $dx \wedge dy$. Since $dx \wedge dy = d(x\,dy)$,

$$\text{Area}(M) = \int_M dx \wedge dy = \int_M d(x\,dy) = \int_{\partial M} x\,dy \quad \text{(Stokes/Green).}$$

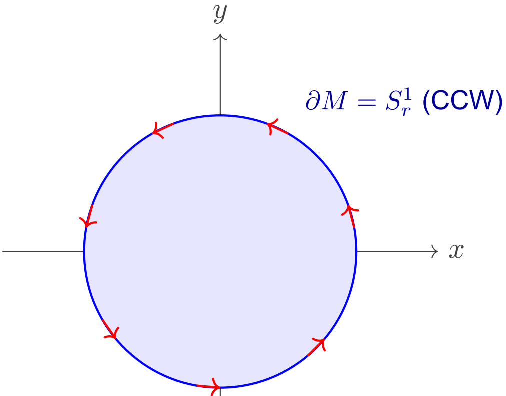
Parametrization

Parametrization and Computation

Parametrize ∂M by $\gamma(\phi) = (r\cos\phi, r\sin\phi)$, $\phi \in [0, 2\pi]$. Then $x(\phi) = r\cos\phi$, $y(\phi) = r\sin\phi$, so $dy = r\cos\phi\, d\phi$ and

$$\begin{aligned}\int_{\partial M} x\, dy &= \int_0^{2\pi} \big(r\cos\phi\big)\big(r\cos\phi\, d\phi\big)\\ &= r^2 \int_0^{2\pi} \cos^2\phi\, d\phi = r^2 \cdot \pi\\ &= \pi r^2.\end{aligned}$$

Visualization



$$\int_{\partial M} x\,dy = \pi r^2$$

Metric Tensor (Riemannian Metric)

Definition (Metric)

Let M be a smooth manifold. A **metric** on M is a bilinear map

$$g: \mathcal{X}(M) \times \mathcal{X}(M) \longrightarrow \mathcal{F}(M),$$

satisfying, for all $X, Y \in \mathcal{X}(M)$:

- *Symmetry:* $g(X, Y) = g(Y, X)$.
- *Positive–definiteness:* $g(X, X) \geq 0$ in $\mathcal{F}(M)$, and $g(X, X) = 0$ iff $X = 0$.

Riemannian manifold

Let M be a smooth manifold and $g: \mathcal{X}(M) \times \mathcal{X}(M) \longrightarrow \mathcal{F}(M)$ be a metric on M. Then the pair (M, g) is a **Riemannian manifold**.

Metric Examples

Examples in Coordinates

Fix local coordinates $(x^1, \ldots, x^D)$ and frame $\{\frac{\partial}{\partial x^i}\}$.

$$g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = g_{ij} \in \mathcal{F}(M), \qquad g = \sum_{i,j=1}^{D} g_{ij}\, dx^i \otimes dx^j,$$

with the matrix field $\underline{g} = (g_{ij})_{1 \le i,j \le D}$ pointwise symmetric positive–definite.

- **Euclidean metric on $\mathbb{R}^D$:** $g_{ij} = \delta_{ij}$, i.e.

$$\underline{g} = I_D = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}, \quad g = \sum_{i=1}^{D} dx^i \otimes dx^i.$$

Metric Examples

General metric on $\mathbb{R}^D$:

- $\underline{g}(x) = (g_{ij}(x))$ with $g_{ij} = g_{ji}$ and $\underline{g}(x)$ positive–definite for each x.

Historical note:

Bernhard Riemann (1826–1866) introduced Riemannian metrics, founding the intrinsic geometry of manifolds and influencing modern differential geometry and physics.

Musical Isomorphisms

Flat, Sharp and Dual Metric

Given a Riemannian metric g, the **flat** operator

$$\flat : \mathcal{X}(M) \to \Omega^1(M), \qquad (\flat X)(Y) = g(X, Y),$$

is a $C^\infty(M)$–isomorphism. Its inverse is the **sharp** operator

$$\sharp = \flat^{-1} : \Omega^1(M) \to \mathcal{X}(M).$$

They induce a metric on 1–forms,

$$g^* : \Omega^1(M) \times \Omega^1(M) \to \mathcal{F}(M), \qquad g^*(\omega, \eta) = \omega(\sharp\eta).$$

Musical Isomorphisms

Flat/Sharp and Dual Metric

In coordinates, with $g = \sum g_{ij}\, dx^i \otimes dx^j$ and inverse matrix (g^{ij}):

$$g^* = \sum_{i,j=1}^{D} g^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}.$$

$$\mathcal{X}(M) \underset{\sharp = \flat^{-1}}{\overset{\flat}{\longleftrightarrow}} \Omega^1(M)$$

$$g(X,Y) = (\flat X)(Y), \quad g^*(\omega,\eta) = \omega(\sharp\eta).$$

Pullback Metric on a Submanifold

Definition (Induced Metric)

Let (M, g) be Riemannian and $S \subseteq M$ a smooth submanifold with embedding $\iota : S \hookrightarrow M$, $\iota(s) = s$. The **induced (pullback) metric** on S is

$$\iota^* g = \sum_{i,j=1}^{D} (g_{ij} \circ \iota)\, d(x^i \circ \iota) \otimes d(x^j \circ \iota).$$

Interpretation

Express g in the ambient coordinates, then pull back each coefficient and each coordinate differential to S via ι (i.e. *change of variables* to the coordinates on S).

Example

Unit Sphere $S^2 \subset \mathbb{R}^3$

Let $M = B_1(0,0,0) = \{(x,y,z) \mid x^2+y^2+z^2 \leq 1\}$ and $B_1(0,0,0) \subset \mathbb{R}^3$, $S = \partial M = S^2 = \{(x,y,z) \mid x^2+y^2+z^2 = 1\}$ and $g = dx \otimes dx + dy \otimes dy + dz \otimes dz$ (Euclidean).
In spherical coordinates on S^2,

$$x = \sin\theta\cos\phi, \quad y = \sin\theta\sin\phi, \quad z = \cos\theta,$$

with

$$(\theta,\phi) \in (0,\pi) \times (0,2\pi).$$

Compute

$$\begin{aligned} dx &= \cos\theta\cos\phi\, d\theta - \sin\theta\sin\phi\, d\phi, \\ dy &= \cos\theta\sin\phi\, d\theta + \sin\theta\cos\phi\, d\phi, \\ dz &= -\sin\theta\, d\theta. \end{aligned}$$

Example

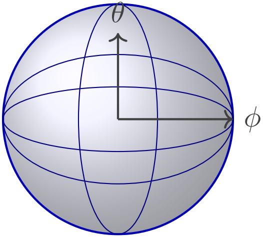
Unit Sphere $S^2 \subset \mathbb{R}^3$

Then
$$\iota^* g = dx^2 + dy^2 + dz^2 = (d\theta)^2 + \sin^2\theta\,(d\phi)^2.$$

Hence, in the chart (θ, ϕ),
$$\underline{h} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}, \qquad \underline{h}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \dfrac{1}{\sin^2\theta} \end{pmatrix},$$

and
$$h^{-1} = \frac{\partial}{\partial\theta} \otimes \frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta} \frac{\partial}{\partial\phi} \otimes \frac{\partial}{\partial\phi}.$$

Visualization



$$\iota^* g = d\theta^2 + \sin^2\theta\, d\phi^2 \text{ on } S^2$$

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Orthonormal Frames

Definition

Let (M, g) be a D–dimensional Riemannian manifold. A frame $(X_1, \ldots, X_D)$ in $\mathcal{X}(M)$ is **orthonormal** if

$$g(X_i, X_j) = \delta_{ij} \qquad (1 \leq i, j \leq D).$$

The dual coframe (or dual basis) $(\omega^1, \ldots, \omega^D)$ in $\Omega^1(M)$ is orthonormal for g^*.

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Examples of Orthonormal Bases

Example 1: Euclidean Space $(\mathbb{R}^D, g = \sum_{i=1}^{D} dx^i \otimes dx^i)$

Consider the standard Cartesian coordinates $(x^1, \ldots, x^D)$ on $\mathbb{R}^D$. The standard basis of vector fields is

$$\left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \ldots, \frac{\partial}{\partial x^D}\right) \in \mathcal{X}(\mathbb{R}^D),$$

and the corresponding dual basis of 1–forms is

$$(dx^1, dx^2, \ldots, dx^D) \in \Omega^1(\mathbb{R}^D).$$

For this metric,

$$g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \delta_{ij}, \qquad g^*(dx^i, dx^j) = \delta_{ij}.$$

Examples of Orthonormal Bases

Example 1: Euclidean Space $(\mathbb{R}^D, g = \sum_{i=1}^D dx^i \otimes dx^i)$

Hence both sets are orthonormal:

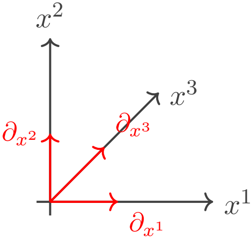
$$g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^i}\right) = 1, \quad g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = 0 \ (i \neq j),$$

and similarly for the dual coframe.

Interpretation:

The Euclidean metric measures inner products as the usual dot product; hence, in these coordinates, vectors and covectors have the same orthonormal structure.

Visualization



Orthonormal frame in $\mathbb{R}^3$

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Example 2: Orthonormal Basis

On the Sphere S^2

On S^2 with the induced metric from $\mathbb{R}^3$,

$$h = d\theta^2 + \sin^2\theta\, d\phi^2,$$

where (θ, ϕ) are the standard spherical coordinates:

$$x = \sin\theta\cos\phi, \qquad y = \sin\theta\sin\phi, \qquad z = \cos\theta.$$

Compute h on the coordinate vector fields:

$$h\left(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta}\right) = 1,$$

$$h\left(\frac{\partial}{\partial\phi}, \frac{\partial}{\partial\phi}\right) = \sin^2\theta,$$

Example 2: Orthonormal Basis

On the Sphere S^2

$$h\left(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\phi}\right) = 0.$$

Hence, the rescaled frame

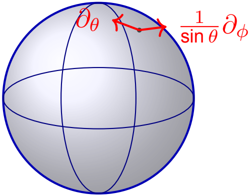
$$\left(\frac{\partial}{\partial\theta},\ \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right)$$

is orthonormal:

$$h\left(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta}\right) = h\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}, \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right) = 1, \quad h\left(\frac{\partial}{\partial\theta}, \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right) = 0.$$

The dual orthonormal coframe is $(d\theta,\ \sin\theta\, d\phi)$.

Example 2: Orthonormal Basis



Orthonormal tangent frame and coframe on S^2

Inner Product on Differential Forms

Definition

Let (M, g) be a D–dimensional Riemannian manifold. The metric g induces an inner product on p–forms:

$$g_p^* : \ \Omega^p(M) \times \Omega^p(M) \to \mathcal{F}(M), \qquad p = 0, 1, \ldots, D,$$

defined recursively:

$$\begin{aligned} g_0^*(f_1, f_2) &= f_1 f_2, \qquad && f_1, f_2 \in \mathcal{F}(M), \\ g_1^*(\omega, \eta) &= g^*(\omega, \eta), && \omega, \eta \in \Omega^1(M), \end{aligned}$$

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Inner Product on p–Forms

Definition

Let (M, g) be a D–dimensional Riemannian manifold. For $p \geq 2$, the metric g induces an inner product on p–forms

$$g_p^* : \Omega^p(M) \times \Omega^p(M) \longrightarrow \mathcal{F}(M),$$

defined on decomposable forms as

$$g_p^*(\omega_1 \wedge \cdots \wedge \omega_p,\ \eta_1 \wedge \cdots \wedge \eta_p) = \begin{vmatrix} g^*(\omega_1, \eta_1) & g^*(\omega_1, \eta_2) & \cdots & g^*(\omega_1, \eta_p) \\ g^*(\omega_2, \eta_1) & g^*(\omega_2, \eta_2) & \cdots & g^*(\omega_2, \eta_p) \\ \vdots & \vdots & \ddots & \vdots \\ g^*(\omega_p, \eta_1) & g^*(\omega_p, \eta_2) & \cdots & g^*(\omega_p, \eta_p) \end{vmatrix}.$$

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Explanation

The determinant encodes all mutual inner products between $\{\omega_i\}$ and $\{\eta_j\}$:

$$g_p^*(\omega_1\wedge\ldots\wedge\omega_p,\eta_1\wedge\ldots\wedge\eta_p)=\sum_{\sigma\in S_p}\mathrm{sgn}(\sigma)\prod_{k=1}^{p}g^*(\omega_k,\eta_{\sigma(k)}),$$

where S_p is the permutation group on $\{1,\ldots,p\}$ and $\mathrm{sgn}(\sigma)$ is the sign of permutation σ. This generalizes the usual Euclidean determinant form of the inner product on wedge products.

$$g_p^*(\omega_1\wedge\cdots\wedge\omega_p,\eta_1\wedge\cdots\wedge\eta_p)=\det[g^*(\omega_i,\eta_j)]_{1\le i,j\le p}$$

$$g_p^*(\omega,\eta)=\sum_{\sigma\in S_p}\mathrm{sgn}(\sigma)\,g^*(\omega_1,\eta_{\sigma(1)})\cdots g^*(\omega_p,\eta_{\sigma(p)}).$$

Example

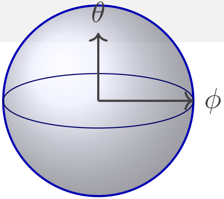
S^2 with $h = d\theta^2 + \sin^2\theta\, d\phi^2$

$$g_2^*(d\theta \wedge d\phi,\ d\theta \wedge d\phi) = \det \begin{pmatrix} g^*(d\theta, d\theta) & g^*(d\theta, d\phi) \\ g^*(d\phi, d\theta) & g^*(d\phi, d\phi) \end{pmatrix}$$

$$= \det \begin{pmatrix} 1 & 0 \\ 0 & \dfrac{1}{\sin^2\theta} \end{pmatrix} = \frac{1}{\sin^2\theta}.$$

Hence

$$g_2^*(\sin\theta\, d\theta \wedge d\phi,\ \sin\theta\, d\theta \wedge d\phi) = \sin^2\theta \cdot \frac{1}{\sin^2\theta} = 1.$$



$$g_2^*(d\theta \wedge d\phi, d\theta \wedge d\phi) = \frac{1}{\sin^2\theta}$$

Volume Form

Definition

Let (M, g) be a D–dimensional Riemannian manifold. Since $\Omega^D(M)$ is 1–dimensional, there exist (up to sign) two D–forms ω and $-\omega$ such that

$$g_D^*(\omega, \omega) = 1.$$

Any such form is a **volume form** of (M, g). If M is oriented, the sign of ω is fixed canonically, and we denote $\mathrm{vol}_g = \omega$.

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Examples

Example

- On $\mathbb{R}^D$ with $g = \sum dx^i \otimes dx^i$:

$$\mathrm{vol}_g = \pm\, dx^1 \wedge dx^2 \wedge \cdots \wedge dx^D.$$

- On S^2 with $h = d\theta^2 + \sin^2\theta\, d\phi^2$:

$$\mathrm{vol}_h = \sin\theta\, d\theta \wedge d\phi.$$

Proposition

Statement: Volume Form from Orthonormal Basis

Let (M, g) be an oriented D–dimensional Riemannian manifold and $e^1, \ldots, e^D$ an orthonormal basis of $\Omega^1(M)$ consistent with the orientation. Then

$$\mathrm{vol}_g = e^1 \wedge e^2 \wedge \cdots \wedge e^D.$$

Examples

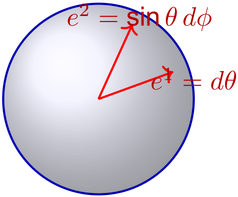
Example

- On $(\mathbb{R}^D, g)$ with standard basis $dx^1, \ldots, dx^D$:

$$\mathrm{vol}_g = dx^1 \wedge dx^2 \wedge \cdots \wedge dx^D.$$

- On S^2 with orthonormal coframe $(d\theta, \sin\theta\, d\phi)$:

$$\mathrm{vol}_h = d\theta \wedge (\sin\theta\, d\phi) = \sin\theta\, d\theta \wedge d\phi.$$



$$\mathrm{vol} = e^1 \wedge e^2$$

Hodge Star Operator

Definition

Let (M, g) be a D–dimensional oriented Riemannian manifold with volume form vol_g. The **Hodge star** is the unique $\mathcal{F}(M)$–linear map

$$\star_p : \Omega^p(M) \to \Omega^{D-p}(M)$$

such that for all $\omega, \eta \in \Omega^p(M)$,

$$g_p^*(\omega, \eta)\,\mathrm{vol}_g = \omega \wedge \star_p \eta.$$

Equivalently, for $\omega \in \Omega^p(M)$, $\eta \in \Omega^{D-p}(M)$,

$$g_{D-p}^*(\star_p \omega, \eta)\,\mathrm{vol}_g = \omega \wedge \eta.$$

Theorem: Properties of the Hodge Star

Statement

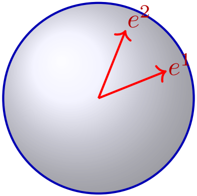
Let (M, g) be an oriented D–dimensional Riemannian manifold and $e^1, \ldots, e^D$ be an oriented orthonormal basis of $\Omega^1(M)$.

1. For an index set $I = \{i_1 < \cdots < i_p\}$, let $e^I = e^{i_1} \wedge \cdots \wedge e^{i_p}$, and J the complementary set of indices. Then
$$\star_p e^I = s\, e^J, \quad s \in \{-1, 1\},$$
where $e^I \wedge e^J = s\, \mathrm{vol}_g = s\, e^1 \wedge \cdots \wedge e^D$.
2. $\star_{D-p} \circ \star_p = (-1)^{p(D-p)}\, \mathrm{id}_{\Omega^p(M)}$.
In particular, if D is odd, $\star_{D-p} \circ \star_p = \mathrm{id}_{\Omega^p(M)}$.

Examples: Hodge Star in $D = 2$

Example: S^2 with orthonormal coframe $(e^1 = d\theta,\ e^2 = \sin\theta\, d\phi)$

$$\star_0\,(1) = e^1 \wedge e^2 = \mathrm{vol},$$
$$\star_1\,(e^1) = e^2, \qquad \star_1(e^2) = -e^1,$$
$$\star_2\,(\mathrm{vol}) = \star_2(e^1 \wedge e^2) = 1.$$



$$\star_1 e^1 = e^2, \quad \star_1 e^2 = -e^1$$

Examples: Hodge Star in $D = 3$

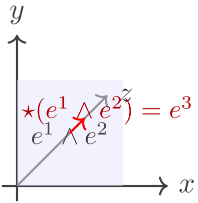
Example: $\mathbb{R}^3$ with orthonormal coframe $(e^1 = dx,\ e^2 = dy,\ e^3 = dz)$

$$\star_0\,(1) = e^1 \wedge e^2 \wedge e^3 = \text{vol},$$

$$\star_1\,(e^1) = e^2 \wedge e^3, \quad \star_1(e^2) = e^3 \wedge e^1, \quad \star_1(e^3) = e^1 \wedge e^2,$$

$$\star_2\,(e^1 \wedge e^2) = e^3, \quad \star_2(e^2 \wedge e^3) = e^1, \quad \star_2(e^3 \wedge e^1) = e^2,$$

$$\star_3\,(e^1 \wedge e^2 \wedge e^3) = 1.$$



Summary

- Area of a disk via Green/Stokes: Area$(\overline{B_r}) = \int_{\partial B_r} x\,dy = \pi r^2$.
- Riemannian metric g gives lengths/angles: $g = \sum g_{ij}\,dx^i \otimes dx^j$, with inverse $g^* = \sum g^{ij}\,\partial_i \otimes \partial_j$.
- Musical isomorphisms $\flat, \sharp$ identify vectors and 1–forms.
- Induced metric on submanifolds: $\iota^* g$; for S^2, $d\theta^2 + \sin^2\theta\,d\phi^2$.
- Orthonormal frames simplify computations (e.g. on S^2: $(\partial_\theta, \frac{1}{\sin\theta}\partial_\phi)$).
- g_p^* defines inner products on all p–forms via determinant of $g^*(\cdot,\cdot)$.

Summary

- Volume form vol_g satisfies $g_D^*(\mathrm{vol}_g, \mathrm{vol}_g) = 1$ and encodes orientation.
- $\mathrm{vol}_g = e^1 \wedge \cdots \wedge e^D$ for any oriented orthonormal coframe.
- Hodge star $\star_p$ maps $\Omega^p(M) \to \Omega^{D-p}(M)$ satisfying $g_p^*(\omega, \eta)\mathrm{vol}_g = \omega \wedge \star_p \eta$.
- Key identity: $\star_{D-p} \circ \star_p = (-1)^{p(D-p)}\mathrm{id}$.

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Thanks