

Combinatorial Mesh Calculus (CMC): Lecture 11

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Coordinate Patch in Polar Form

Example - Setting

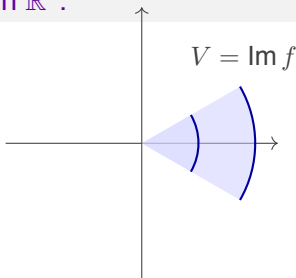
Let $U = (\frac{1}{2}, 1] \times (-\frac{\pi}{6}, \frac{\pi}{6})$ and define

$$f : U \rightarrow \mathbb{R}^2, \quad f(r, \phi) = (r \cos \phi, r \sin \phi).$$

Then the image

$$V = \text{Im } f = \{(r \cos \phi, r \sin \phi) \mid r \in (\frac{1}{2}, 1], \phi \in (-\frac{\pi}{6}, \frac{\pi}{6})\}$$

is an annular sector in $\mathbb{R}^2$.



Exterior Derivative in $\mathbb{R}^3$

Definition

For a smooth manifold $M = \mathbb{R}^3$, the exterior derivative (ED)

$$d_p : \Omega^p M \rightarrow \Omega^{p+1} M$$

satisfies $d_{p+1} \circ d_p = 0$ and the graded Leibniz rule.

Concrete Computations

Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}$.

$$d_0 f = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz.$$

For a 1-form $\omega = f_x dx + f_y dy + f_z dz$,

Example

Concrete Computations

$$\begin{aligned} d_1\omega &= d(f_x) \wedge dx + d(f_y) \wedge dy + d(f_z) \wedge dz \\ &= (f_{zy} - f_{yz}) dy \wedge dz + (f_{xz} - f_{zx}) dz \wedge dx + (f_{yx} - f_{xy}) dx \wedge dy. \end{aligned}$$

Thus $d_1\omega$ corresponds to $\text{curl}(f_x, f_y, f_z)$.

For a 2-form

If $\eta = f_x dy \wedge dz + f_y dz \wedge dx + f_z dx \wedge dy$, then

$$d_2\eta = (f_{x,x} + f_{y,y} + f_{z,z}) dx \wedge dy \wedge dz,$$

which represents the divergence $\nabla \cdot (f_x, f_y, f_z)$.

Differential Operators as ED

In $\mathbb{R}^2$

$$d_0 f = \nabla f = (f_x, f_y),$$

$$d_1(f_x dx + f_y dy) = (f_{yx} - f_{xy}) dx \wedge dy \Rightarrow \text{scalar curl},$$

$$d_1 \circ d_0 = 0 \Rightarrow \text{curl}(\nabla f) = 0.$$

$$\begin{array}{c} \mathcal{F}(\mathbb{R}^2) = \Omega^0 \\ \downarrow d_0 = \nabla \\ \mathcal{X}(\mathbb{R}^2) = \Omega^1 \\ \downarrow d_1 = \text{curl} \\ \Omega^2(\mathbb{R}^2) \end{array}$$

Differential Operators as ED

In $\mathbb{R}^3$

$$d_0 f = \nabla f,$$

$$d_1(f_x dx + f_y dy + f_z dz) = \mathbf{curl}(f_x, f_y, f_z),$$

$$d_2(f_x dy \wedge dz + \dots) = \mathbf{div}(f_x, f_y, f_z),$$

$$d_2 \circ d_1 = 0 \Rightarrow \nabla \cdot (\nabla \times A) = 0.$$

$$\begin{array}{c} \mathcal{F}(\mathbb{R}^3) = \Omega^0 \\ \downarrow d_0 = \nabla \\ \mathcal{X}(\mathbb{R}^3) = \Omega^1 \\ \downarrow d_1 = \mathbf{curl} \\ \Omega^2(\mathbb{R}^3) \\ \downarrow d_2 = \mathbf{div} \\ \Omega^3(\mathbb{R}^3) \end{array}$$

Pullback of Differential Forms

Definition

Let M, N be smooth manifolds and $f \in C^\infty(M, N)$. For a 1-form $\omega \in \Omega^1 N$ with local expression

$$\omega = h_1 dy_1 + \cdots + h_n dy_n, \quad h_i \in \mathcal{F}(N),$$

the pullback $f^*\omega \in \Omega^1 M$ is defined by

$$f^*\omega = (h_1 \circ f) d(y_1 \circ f) + \cdots + (h_n \circ f) d(y_n \circ f).$$

It extends naturally to exterior powers: $f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta$.

Pullback of Differential Forms

Example

Let $M = N = \mathbb{R}^2$, $f(r, \phi) = (r \cos \phi, r \sin \phi)$, and $\omega = -y dx + x dy$. Then

$$\begin{aligned} f^* \omega &= - (r \sin \phi) d(r \cos \phi) + (r \cos \phi) d(r \sin \phi) \\ &= - (r \sin \phi) (\cos \phi dr - r \sin \phi d\phi) \\ &\quad + (r \cos \phi) (\sin \phi dr + r \cos \phi d\phi) \\ &= r^2 d\phi. \end{aligned}$$

Compatibility of Pullback and ED

Theorem (Compatibility of Pullback and Exterior Derivative)

Let M, N be smooth manifolds and $f \in C^\infty(M, N)$ a smooth map. For each degree $p \in \mathbb{N}$, the pullback operator

$$f_p^* : \Omega^p(N) \rightarrow \Omega^p(M)$$

commutes with the exterior derivative, i.e.

$$d_p^M \circ f_p^* = f_{p+1}^* \circ d_p^N.$$

In other words, taking the pullback of a form and then differentiating gives the same result as differentiating first and then taking the pullback:

$$f^*(d\omega) = d(f^*\omega), \quad \forall \omega \in \Omega^p(N).$$

This shows that the exterior derivative is a *natural operator* with respect to smooth maps between manifolds.

Example

$$\begin{array}{ccc}
 \Omega^p(N) & \xrightarrow{d_p^N} & \Omega^{p+1}(N) \\
 f_p^* \downarrow & & \downarrow f_{p+1}^* \\
 \Omega^p(M) & \xrightarrow{d_p^M} & \Omega^{p+1}(M)
 \end{array}$$

Example

Let $M = N = \mathbb{R}^2$, $f(r, \phi) = (r \cos \phi, r \sin \phi)$, $\omega = dx \wedge dy$:

$$f^* dx = \cos \phi \, dr - r \sin \phi \, d\phi,$$

$$f^* dy = \sin \phi \, dr + r \cos \phi \, d\phi,$$

$$f^*(dx \wedge dy) = (f^* dx) \wedge (f^* dy) = r \, dr \wedge d\phi.$$

Definition

Let M be a smooth manifold and $N \subseteq M$ a submanifold with inclusion map

$$\iota_N : N \hookrightarrow M, \quad \iota_N(x) = x.$$

The **trace** (restriction) of forms is the pullback

$$\text{tr}_N = \iota_N^* : \Omega^\bullet M \rightarrow \Omega^\bullet N.$$

If $N \subseteq \partial M$, this represents the restriction of forms to the boundary.

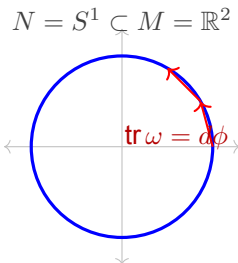
Example

Example

Let $M = \mathbb{R}^2$, $N = S^1 \subseteq M$, and $\omega = x dy - y dx$. The inclusion $\iota_{S^1} : S^1 \rightarrow \mathbb{R}^2$ gives

$$\begin{aligned}\text{tr}_{S^1} \omega &= (x dy - y dx)|_{S^1} \\ &= (r \cos \phi)(r \cos \phi d\phi) - (r \sin \phi)(-r \sin \phi d\phi) = r^2 d\phi,\end{aligned}$$

and for $r = 1$, $\text{tr}_{S^1} \omega = d\phi$.



Orientation of Smooth Manifolds

Definition

Let M be a smooth manifold of dimension D . Since the space of top-degree forms $\Omega^D M$ is locally 1-dimensional, every element $\omega \in \Omega^D M$ can be written locally as

$$\omega = f dx_1 \wedge dx_2 \wedge \cdots \wedge dx_D, \quad f \in \mathcal{F}(M).$$

If $f(x) \neq 0$ for all $x \in M$, then ω is said to be a **nonvanishing D -form**, and M is called **orientable**. A choice of such an ω (and identifying all positive scalar multiples of ω as equivalent) defines an **orientation** of M .

If M is connected and orientable, it admits exactly two possible orientations, represented by ω and $-\omega$. If M is disconnected, each connected component may be oriented independently.

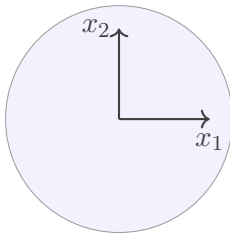
Orientation of Smooth Manifolds

Intuitively, an orientation distinguishes between the two possible “directions of measurement” on M : one associated with ω (positive orientation) and the other with $-\omega$ (negative orientation).

Example

$\mathbb{R}^D$ is orientable with orientation

$$\omega = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_D, \quad \text{or its opposite } -dx_1 \wedge \cdots \wedge dx_D.$$



$\omega = dx_1 \wedge dx_2$ gives positive orientation

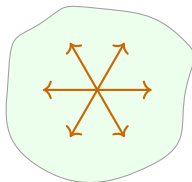
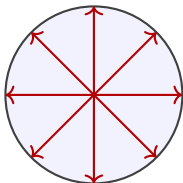
Outward-Pointing Vector Fields

Definition

Let $M \subseteq \mathbb{R}^D$ be a D -dimensional manifold with boundary. A vector field $X \in \mathcal{X}(M)$ is called **outward-pointing** on ∂M if near every boundary point it locally points outside M .

Examples

- $M = \overline{B}_1(0,0) \subseteq \mathbb{R}^2$, $X = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$;
- $M = \overline{B}_1(0,0,0) \subseteq \mathbb{R}^3$, $X = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z}$.



The Induced Orientation on Boundaries

Theorem (Induced Orientation)

Let M be an orientable D -manifold with boundary $S = \partial M$ and orientation form $\omega \in \Omega^D M$. If $X \in \mathcal{X}(M)$ is outward-pointing, then

$$\omega_S = \text{tr}_{\partial M}(\iota_X \omega) \in \Omega^{D-1}(S)$$

defines an orientation form on S , called the **induced orientation**.

Example

Example 1: Half-Plane

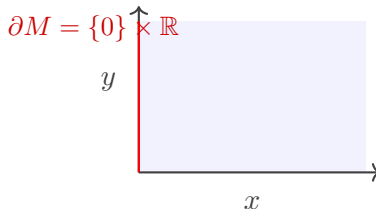
Let $M = [0, \infty) \times \mathbb{R}$, $S = \{0\} \times \mathbb{R}$,

$$X = -\frac{\partial}{\partial x}, \quad \omega = dx \wedge dy.$$

Then

$$\iota_X(dx \wedge dy) = \iota_{-\partial_x}(dx \wedge dy) = -\iota_{\partial_x}(dx \wedge dy) = -dy.$$

Thus $\omega_S = -dy$: the boundary inherits the leftward orientation.



Induced Orientation in 2D and 3D Balls

Example 2: Disk in $\mathbb{R}^2$

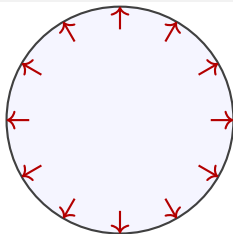
$$M = B_1(0,0), \partial M = S^1, X = x\partial_x + y\partial_y, \quad \omega_M = dx \wedge dy.$$

$$\text{Then } \iota_X \omega_M = \iota_{x\partial_x + y\partial_y} (dx \wedge dy) = x dy - y dx.$$

In polar coordinates $x = r \cos \phi$, $y = r \sin \phi$, so

$$x dy - y dx = r^2 d\phi \quad \Rightarrow \quad \text{tr}_{S^1}(\iota_X \omega_M) = d\phi.$$

Hence, S^1 inherits counterclockwise orientation.



3D Ball: Induced Orientation on the Sphere

Example 3: Ball in $\mathbb{R}^3$

$$M = B_1(0, 0, 0), \partial M = S^2,$$

$$X = x\partial_x + y\partial_y + z\partial_z, \quad \omega_M = dx \wedge dy \wedge dz.$$

Then

$$\iota_X \omega_M = x dy \wedge dz + y dz \wedge dx + z dx \wedge dy.$$

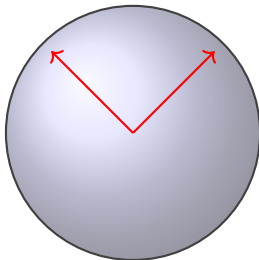
In spherical coordinates

$x = r \sin \theta \cos \phi$, $y = r \sin \theta \sin \phi$, $z = r \cos \theta$, we have

$$\text{tr}_{S^2}(\iota_X \omega_M) = r^2 \sin \theta d\theta \wedge d\phi.$$

This 2-form defines the standard orientation of S^2 .

Visualization



$$\omega_S = \sin \theta \, d\theta \wedge d\phi$$

Spherical Coordinates

Spherical chart on S^2 (radius 1)

Standard spherical coordinates

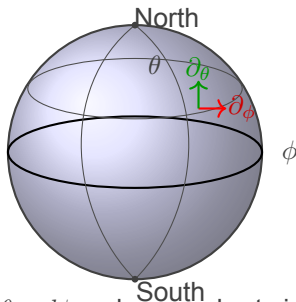
$$x = \sin \theta \cos \phi, \quad y = \sin \theta \sin \phi, \quad z = \cos \theta, \quad \theta \in (0, \pi), \quad \phi \in (0, 2\pi)$$

yield the induced orientation 2-form

$$\omega_{S^2} = \sin \theta \, d\theta \wedge d\phi.$$

Here $d\theta \wedge d\phi$ is ordered so that $(\partial_\theta, \partial_\phi)$ agrees with the outward normal orientation. $\sin \theta$ is the Jacobian density (area scale factor) of the chart; it vanishes only at the poles $\theta = 0, \pi$, where this chart is *singular* (coordinate singularity, not a geometric one).

Visualization



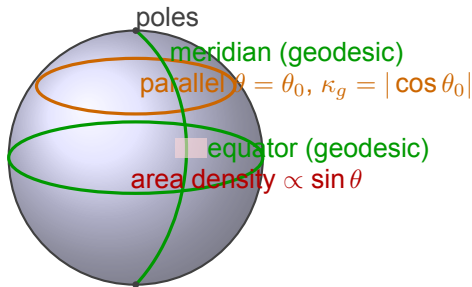
$\omega_{S^2} = \sin \theta \, d\theta \wedge d\phi$; poles are chart singularities.

Curvature Intuition on S^2 : Meridians vs Parallels

Geodesics and geodesic curvature (unit sphere)

- **Meridians** ($\phi = \text{const}$) and **equator** ($\theta = \pi/2$) are *great circles* $\Rightarrow$ geodesics with geodesic curvature $\kappa_g = 0$.
- **Parallels** ($\theta = \text{const} \neq \pi/2$) are not geodesics; their geodesic curvature is $\kappa_g = |\cos \theta|$ (nonzero away from the equator).
- The area element (orientation form) is $\omega_{S^2} = \sin \theta \, d\theta \wedge d\phi$: bands near the equator ($\theta \approx \pi/2$) have greater area density; near the poles ($\theta \rightarrow 0, \pi$) the density vanishes.

Visualization



Summary

The θ, ϕ chart encodes orientation and area via $\sin \theta$, is singular at the poles, and separates geodesic directions (great circles) from curved parallels (nonzero κ_g).

Compact Manifolds and Integration

Definition

A manifold $M \subseteq \mathbb{R}^D$ is **compact** if it is closed and bounded, i.e. it lies entirely inside some finite ball $B_R(0)$.

Examples

- Closed D -bricks and their boundaries,
- Closed D -balls $\overline{B}_1(0)$ and spheres S^{D-1} .

Smooth Oriented D -Manifolds

For compact, oriented M , integration is a linear map

$$\int_M : \Omega^D M \rightarrow \mathbb{R}$$

satisfying:

1. **Additivity:** If $M = M_1 \cup M_2$ with same orientation,

$$\int_M \omega = \int_{M_1} \text{tr}_{M_1} \omega + \int_{M_2} \text{tr}_{M_2} \omega.$$

2. **Change of Variables:** If $\phi : M \rightarrow N$ is an orientation-preserving diffeomorphism,

$$\int_N \omega = \int_M \phi^* \omega.$$

Smooth Oriented D -Manifolds

3 Stokes–Cartan Theorem:

$$\int_M d\omega = \int_{\partial M} \text{tr}_{\partial M} \omega.$$

4 Zero–Dimensional Case: If $M = \{x\}$ with orientation

$$\epsilon = \pm 1, \quad \int_{\{x\}} f = \epsilon f(x).$$

Example 1: ($D = 1$)

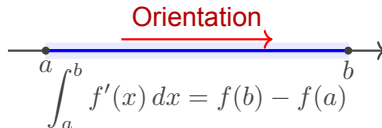
Statement: Newton–Leibniz Theorem

Let $M = [a, b] \subset \mathbb{R}$, with boundary $\partial M = \{a, b\}$ oriented as $a \mapsto b$.
For $f \in C^\infty(M)$,

$$\omega = df = f'(x) dx.$$

Then, by Stokes–Cartan:

$$\int_{[a,b]} df = \int_{\partial[a,b]} \text{tr } f = f(b) - f(a).$$



Orientation

$$\int_a^b f'(x) dx = f(b) - f(a)$$

Example 2: ($D = 2$, Curve in $\mathbb{R}^2$)

Statement: Gradient Theorem

Let $\gamma : [a, b] \rightarrow \mathbb{R}^2$ be a smooth oriented 1-manifold with boundary $\partial\gamma = \{\gamma(a), \gamma(b)\}$, and $f \in C^\infty(\mathbb{R}^2)$. The 1-form

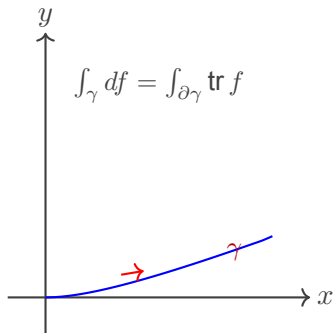
$$\omega = df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy.$$

Then

$$\int_{\gamma} df = \int_{\partial\gamma} \text{tr}_{\partial\gamma} f = f(\gamma(b)) - f(\gamma(a)).$$

Thus, the **gradient theorem** is the 2-dimensional instance of Stokes–Cartan.

Visualization



Example 3: ($D = 2$)

Statement: Green's Theorem

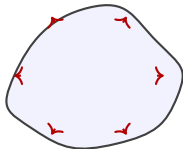
Let $M \subset \mathbb{R}^2$ be a compact oriented 2-manifold with boundary ∂M (positively oriented). For a 1-form $\omega = P dx + Q dy$,

$$d\omega = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy.$$

Then by Stokes–Cartan,

$$\int_M d\omega = \int_{\partial M} \text{tr}_{\partial M} \omega.$$

This is **Green's theorem** in exterior-form form: the flux of $d\omega$ across M equals the trace of ω on ∂M .



Example 4: $D = 3$ Statement (Stokes–Cartan for a 1–manifold in $\mathbb{R}^3$)

Let $\gamma : [a, b] \rightarrow \mathbb{R}^3$ be a smooth oriented curve (a compact 1–manifold $M = \gamma([a, b])$ with $\partial M = \{\gamma(a), \gamma(b)\}$ and orientation from a to b). For $f \in \mathcal{F}(\mathbb{R}^3) = C^\infty(\mathbb{R}^3)$, the 1–form

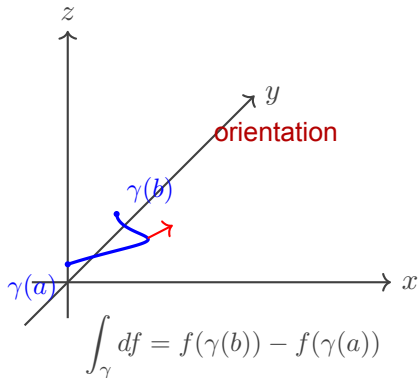
$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz.$$

Then, by Stokes–Cartan on the 1–manifold M ,

$$\int_{\gamma} df = \int_{\partial\gamma} \text{tr}_{\partial\gamma} f = f(\gamma(b)) - f(\gamma(a))$$

(i.e. the *line integral of the exact form df* along the space curve equals the trace of f on the boundary points).

Visualization



Example 5: ($D = 3$)

Statement: Kelvin–Stokes Theorem

Let $M \subset \mathbb{R}^3$ be a smooth oriented 2-manifold with boundary ∂M .

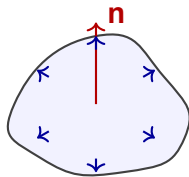
For a 1-form $\omega = A dx + B dy + C dz$,

$$d\omega = \left(\frac{\partial C}{\partial y} - \frac{\partial B}{\partial z} \right) dy \wedge dz + \left(\frac{\partial A}{\partial z} - \frac{\partial C}{\partial x} \right) dz \wedge dx + \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y} \right) dx \wedge dy.$$

Then by Stokes–Cartan:

$$\int_M d\omega = \int_{\partial M} \text{tr}_{\partial M} \omega.$$

This is the **Kelvin–Stokes theorem** in exterior form notation.



Example 6: ($D = 3$)

Statement: Gauss Divergence Theorem

Let $M \subset \mathbb{R}^3$ be a compact oriented 3-manifold with boundary ∂M . For a 2-form

$$\omega = P \, dy \wedge dz + Q \, dz \wedge dx + R \, dx \wedge dy,$$

we have

$$d\omega = (\partial_x P + \partial_y Q + \partial_z R) \, dx \wedge dy \wedge dz.$$

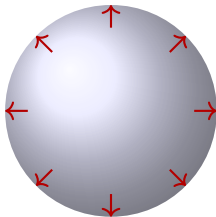
Then, by Stokes–Cartan:

$$\int_M d\omega = \int_{\partial M} \text{tr}_{\partial M} \omega.$$

This is the **Gauss (Divergence) theorem** in the language of differential forms.

Visualization

outward orientation



$$\int_M d\omega = \int_{\partial M} \text{tr}_{\partial M} \omega$$

Summary: Stokes–Cartan in all Dim.

Unified Framework

For any compact oriented smooth D –manifold M with boundary ∂M :

$$\int_M d\omega = \int_{\partial M} \text{tr}_{\partial M} \omega.$$

All classical theorems follow as special cases:

Dim.	Differential Form	Result
1	df	Newton–Leibniz theorem
2	df on γ	Gradient theorem
2	$d(P dx + Q dy)$	Green’s theorem
3	$d(A dx + B dy + C dz)$	Kelvin–Stokes theorem
3	$d(P dy \wedge dz + Q dz \wedge dx + R dx \wedge dy)$	Gauss divergence theorem

Part I: Lecture Summary

Summary

- The exterior derivative $d_p : \Omega^p M \rightarrow \Omega^{p+1} M$ generalizes grad, curl, and divergence.
- In $\mathbb{R}^2$: $d_0 = \text{grad}$, $d_1 = \text{scalar curl}$; in $\mathbb{R}^3$: $d_0 = \text{grad}$, $d_1 = \text{curl}$, $d_2 = \text{div}$.
- Pullback f^* transfers forms from N to M compatibly:
 $d \circ f^* = f^* \circ d$.
- Trace $\text{tr}_N = \iota_N^*$ restricts forms to submanifolds or boundaries.
- These constructions make differential forms coordinate-independent tools for calculus on manifolds.

Part II: Lecture Summary

Summary

- Orientation is given by a nonvanishing top-degree form $\omega \in \Omega^D M$.
- Outward vector fields define induced orientation on ∂M via $\omega_S = \text{tr}_{\partial M}(\iota_X \omega)$.
- Integration of differential forms generalizes classical calculus results to manifolds.
- Stokes–Cartan theorem unifies Newton–Leibniz, Green, Kelvin–Stokes, and Gauss divergence theorems.
- Compact oriented manifolds admit consistent integration respecting additivity and diffeomorphism invariance.

Thanks

