Combinatorial Mesh Calculus (CMC): Lecture 9

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Euclidean Balls and Spheres

Definition.

Let $n \in \mathbb{N}$, $r \in \mathbb{R}^+$, $\vec{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$.

Open ball: $B_r(\vec{x}) := \left\{\vec{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n \mid \sum_{i=1}^{n} (x_i - y_i)^2 < r^2\right\}$.

Closed ball: $\overline{B_r}(\vec{x}) := \left\{\vec{y} \in \mathbb{R}^n \mid \sum_{i=1}^{n} (x_i - y_i)^2 \leq r^2\right\}$.

Sphere: $S_r(\vec{x}) := \left\{\vec{y} \in \mathbb{R}^n \mid \sum_{i=1}^{n} (x_i - y_i)^2 = r^2\right\}$.

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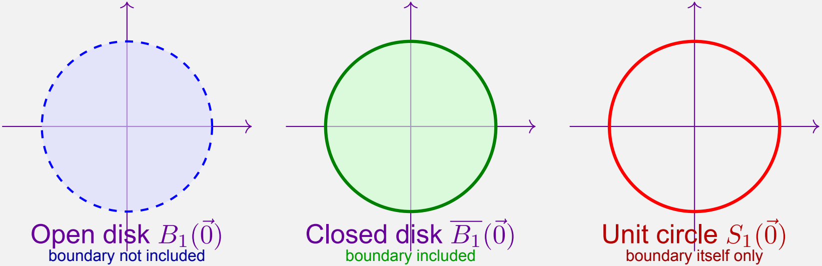
Unit Examples at the Origin

$r = 1,\ \vec{x} = \vec{0}$

Case $n = 1$:

$$B_1(0) = (-1, 1), \quad \overline{B_1}(0) = [-1, 1], \quad S_1(0) = \{-1, 1\}.$$

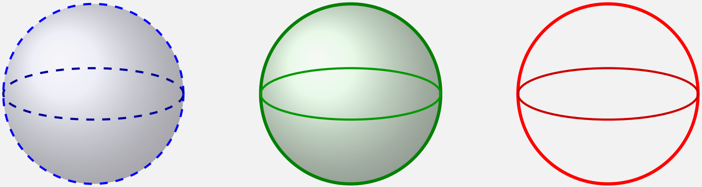
Case $n = 2$: (open unit disk, closed unit disk, unit circle)



Unit Examples at the Origin

Case $n = 3$:

(open unit ball, closed unit ball, unit sphere)



Open ball $B_1(\vec{0})$
boundary not included

Closed ball $\overline{B_1}(\vec{0})$
boundary included

Unit sphere $S_1(\vec{0})$
boundary itself only

Open/Closed n-Bricks

Parallelotopes

Let $n \in \mathbb{N}$ and $a_i < b_i$ for $i = 1, \dots, n$.
Open n–brick: $(a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n)$,
Closed n–brick: $[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$,
and Boundary:

$$
\begin{aligned}
\partial\big([a_1, b_1] \times \cdots \times [a_n, b_n]\big) = & \bigcup_{k=1}^{n} \big([a_1, b_1] \times \cdots \times [a_{k-1}, b_{k-1}] \times \{a_k, b_k\} \\
& \times [a_{k+1}, b_{k+1}] \times \cdots \times [a_n, b_n]\big) \\
= & \Big(\{a_1, b_1\} \times [a_2, b_2] \times \cdots \times [a_n, b_n]\Big) \\
& \cup \Big([a_1, b_1] \times \{a_2, b_2\} \times \cdots \times [a_n, b_n]\Big) \\
& \cup \cdots \cup \Big([a_1, b_1] \times [a_2, b_2] \times \cdots \times \{a_n, b_n\}\Big)
\end{aligned}
$$

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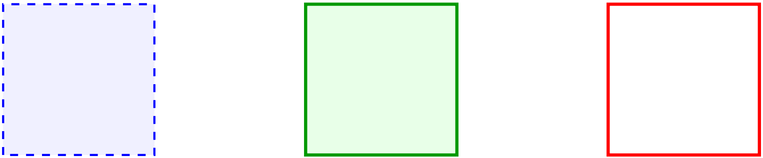
Open and Closed 2D Bricks

Case $n = 2$: unit brick $(0,1)^2$ and $[0,1]^2$ (Squares/Rectangles)

- Open square $(0,1)^2$: interior points only (boundary excluded).
- Closed square $[0,1]^2$: includes all boundary edges.
- Boundary $\partial[0,1]^2$: the four edges forming the perimeter.



Open brick $(0,1)^2$
boundary not included

Closed brick $[0,1]^2$
boundary included

Boundary $\partial[0,1]^2$
four edges only

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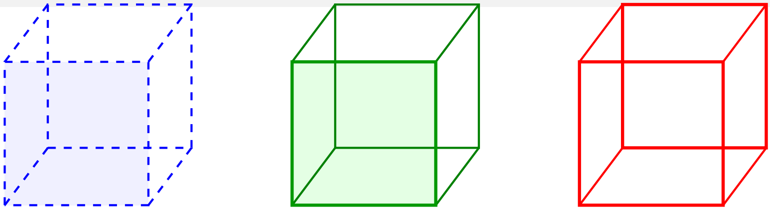
Open and Closed 3D Bricks

Case $n = 3$: unit brick $(0,1)^3$ and $[0,1]^3$

- Open cube $(0,1)^3$: interior points only (no faces).
- Closed cube $[0,1]^3$: includes interior and all 6 faces.
- Boundary $\partial[0,1]^3$: the 6 faces listed below:

$$\{0,1\}\times[0,1]\times[0,1], \quad [0,1]\times\{0,1\}\times[0,1], \quad [0,1]\times[0,1]\times\{0,1\}.$$



Open cube $(0,1)^3$

faces not included

Closed cube $[0,1]^3$

faces included

Boundary $\partial[0,1]^3$

six faces only

Interior and Boundary Points

Definition

Let $n \in \mathbb{N}$, and let $M \subseteq \mathbb{R}^n$.

1. Interior Point

A point $x \in M$ is interior to M if there exists an open ball (equivalently, an open brick) U such that $x \in U \subseteq M$.

2. Boundary Point

A point $x \in \mathbb{R}^n$ is a boundary point of M if for every open neighborhood U containing x, the following two conditions hold:

$$U \cap M \neq \varnothing \quad \text{and} \quad U \cap (\mathbb{R}^n \setminus M) \neq \varnothing.$$

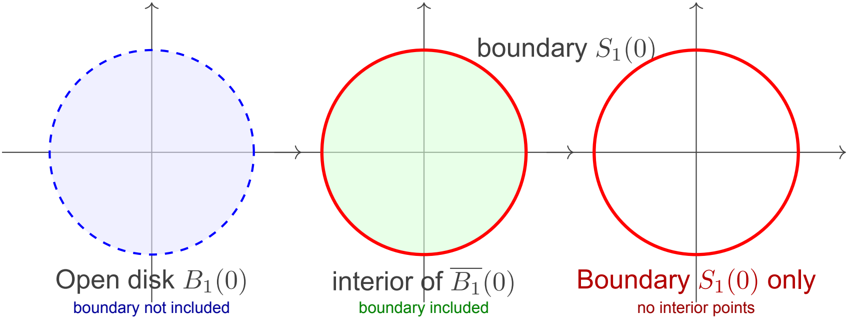
Interior and Boundary Points

Examples in $\mathbb{R}^2$.

$$M_1 = B_1(0,0) = \{(x,y) \mid x^2 + y^2 < 1\}: \text{ every point is interior.}$$

$$M_2 = \overline{B_1}(0,0): \text{ interior points are those with } x^2 + y^2 < 1,$$

$$\text{boundary points are those on the unit circle } x^2 + y^2 = 1.$$



Open and Closed Sets in $\mathbb{R}^n$

Definitions. Let $M \subseteq \mathbb{R}^n$.

$$\begin{aligned} M \text{ is } \textit{open} &\iff \text{every } x \in M \text{ is interior.} \\ M \text{ is } \textit{closed} &\iff \mathbb{R}^n \setminus M \text{ is open.} \end{aligned}$$

Basic examples and reasons.

- $B_r(x)$ is open: for each $y \in B_r(x)$ take a smaller ball $B_\varepsilon(y) \subset B_r(x)$.
- $\overline{B_r}(x)$ is closed: its complement is open (distance to x is continuous; $\{d > r\}$ is open).
- Open n-bricks are open (product of open intervals); closed n-bricks are closed (finite intersection of closed half-spaces).
- $\mathbb{R}^n$ and $\varnothing$ are both open and closed: complements are $\varnothing$ and $\mathbb{R}^n$, respectively, which are open.

Interior and Closure of a Set

Definitions.

For $M \subseteq \mathbb{R}^n$:

int(M) :=the set of interior points of M (largest open subset of M),

$\overline{M}$:=the smallest closed set containing M

(intersection of all closed supersets).

Example.

$M = [0, 1) \subseteq \mathbb{R}$.

$$\text{int}(M) = (0, 1), \qquad \overline{M} = [0, 1].$$

Relative (Subspace) Topology

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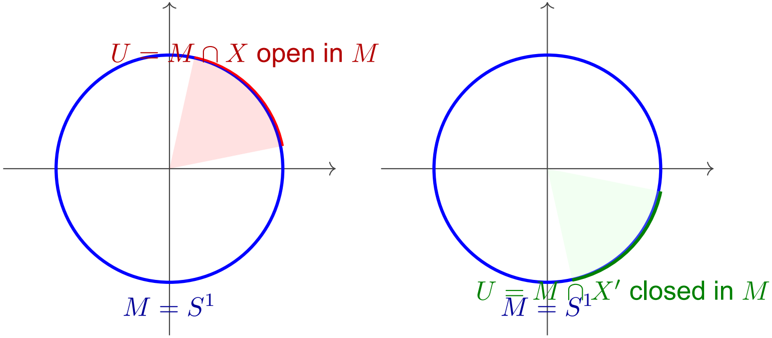
Definition.

Let M be a subset of $\mathbb{R}^n$ and U be a subset of M. We define the relative topology on M by setting its open and closed sets as follows:

$$\begin{aligned} U \text{ is } \textit{open in } M \quad &\iff \quad \exists \text{ an open set } X \subseteq \mathbb{R}^n \\ &\qquad \text{such that } U = M \cap X. \\ U \text{ is } \textit{closed in } M \quad &\iff \quad \exists \text{ a closed set } X \subseteq \mathbb{R}^n \\ &\qquad \text{such that } U = M \cap X. \end{aligned}$$

Relative (Subspace) Topology

Example ($n = 2$). $M = S^1 = \{(x, y) \mid x^2 + y^2 = 1\}$.



Similarly: If X is closed (e.g. intersection with a closed half-plane), then $U = M \cap X$ is closed in M.

Note. Whether M itself is open/closed in $\mathbb{R}^n$ is independent from being open/closed in M; any M is both open and closed in the relative topology on M.

Continuous vs. Smooth

Smoothness on Open/Closed Domains

Continuous means: small changes in input yield small changes in output (no derivatives required). **Smooth** (infinitely differentiable) means: all partial derivatives of all orders exist and are continuous.

Definition (Smooth map on an open domain).

Let $m, n \in \mathbb{N}$, $U \subseteq \mathbb{R}^m$ be open, $V \subseteq \mathbb{R}^n$, and $f = (f_1, \ldots, f_n) : U \to V$. We say f is *smooth* (write $f \in C^\infty(U, V)$) if for every $k \in \mathbb{N}^+$, for all choices $i_1, \ldots, i_k \in \{1, \ldots, m\}$ and each $j \in \{1, \ldots, n\}$, the mixed partial

$$\frac{\partial^k f_j}{\partial x_{i_1} \cdots \partial x_{i_k}}(x)$$

exists for every $x \in U$.

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Smoothness on a closed domain

Definition

If $U \subseteq \mathbb{R}^m$ is *not open* (e.g. closed), we say $f : U \to V$ is smooth if there exist an open set $\widehat{U} \subseteq \mathbb{R}^m$ with $U \subseteq \widehat{U}$ and a smooth map $\widehat{f} \in C^\infty(\widehat{U}, V)$ such that $\widehat{f}\restriction_U = f$.

$$\begin{tikzcd} U \arrow[r, "\text{incl}"] \arrow[dr, "f"'] & \widehat{U} \arrow[d, "\widehat{f} \in C^\infty(\widehat{U}, V)"] \\ & V \end{tikzcd}$$

Examples

- **Elementary Functions:** The following are smooth (C^∞ on their natural domains):
  - **Polynomials** (on all of $\mathbb{R}$).
  - **Trigonometric Functions:** $\sin(x)$ and $\cos(x)$ (on all of $\mathbb{R}$).
  - **Exponential Function:** $\exp(x)$ (on all of $\mathbb{R}$).
  - **Logarithm:** $\ln(x)$ (on its domain, $(0, \infty)$).
  - **Sums, Products, and Quotients** (where the denominator is non-zero).
  - **Compositions:** If f and g are smooth, then $f \circ g$ is smooth.
- **A Function That Fails C^1 at a Point:**
  Consider the function $f(x) = |x|^{1/3}$.
- **The Issue:** This function is not C^1 at $x = 0$.
  The derivative is $f'(x) = \frac{1}{3}|x|^{-2/3} \cdot \operatorname{sgn}(x)$.
- **Conclusion:** $\lim_{x\to 0} |f'(x)| = \lim_{x \to 0} \frac{1}{3|x|^{2/3}} = \infty$. The derivative is not defined (it "blows up") at $x = 0$, meaning $f(x)$ is not differentiable at this point, and thus cannot be C^1 there.

Differential / Jacobian Matrix

Definition. Let $m, n \in \mathbb{N}$, $U \subseteq \mathbb{R}^m$ open, $f = (f_1, \dots, f_n) \in C^\infty(U, \mathbb{R}^n)$ and $x = (x_1, \dots, x_m) \in U$. The *Jacobian (differential) at* x is the $n \times m$ matrix

$$D_x f = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \cdots & \frac{\partial f_1}{\partial x_m}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1}(x) & \cdots & \frac{\partial f_n}{\partial x_m}(x) \end{pmatrix}.$$

It represents the best linear approximation to f at x.

Example (Polar coordinates). $f : [0,1] \times [0, 2\pi] \to \overline{B_1(0,0)}$, $f(r, \phi) = (r \cos \phi,\ r \sin \phi)$. Then

$$\frac{\partial f}{\partial r}(r, \phi) = (\cos \phi,\ \sin \phi), \qquad \frac{\partial f}{\partial \phi}(r, \phi) = (-r \sin \phi,\ r \cos \phi).$$

Hence

$$D_{(r,\phi)} f = \begin{pmatrix} \cos \phi & -r \sin \phi \\ \sin \phi & r \cos \phi \end{pmatrix}, \qquad \det D_{(r,\phi)} f = r.$$

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Inverse Function Theorem (IFT)

Theorem

Let $m \in \mathbb{N}$, $U \subseteq \mathbb{R}^m$ open, and $f \in C^\infty(U, \mathbb{R}^m)$. If at $x_0 \in U$ the Jacobian $D_{x_0} f$ is invertible (i.e. $\det D_{x_0} f \neq 0$), then there exist open neighborhoods $x_0 \in U_0 \subseteq U$ and $f(x_0) \in V_0 \subseteq \mathbb{R}^m$ such that

$$f\restriction_{U_0} : U_0 \to V_0$$

is a *diffeomorphism* (bijective, smooth inverse).

Proof (complete outline with key steps).

- *Reduction:* By translation and linear change of variables (compose with $D_{x_0} f^{-1}$) assume $x_0 = 0$, $f(0) = 0$, $D_0 f = I$.
- *Fixed-point map:* For y near 0, define $T_y(x) = x - (f(x) - y)$. Then $T_y(0) = y$ and $DT_y(0) = I - Df(0) = 0$.

Proof

- *Contraction:* Using continuity of Df at 0, choose a small ball where $\|Df(x) - I\| \leq \frac{1}{2}$, so T_y is a contraction on that ball for all y in a small ball. By Banach's fixed-point theorem, T_y has a unique fixed point x, i.e. $f(x) = y$.
- *Regularity:* The fixed point depends smoothly on y (by implicit function theorem/contractive mapping with parameters), giving $f^{-1} \in C^\infty(V_0, U_0)$.

Example

Polar-annulus example.

Let $U = (\frac{1}{2}, 1) \times (0, 2\pi)$ and define

$$f(r, \phi) = (r \cos \phi,\ r \sin \phi).$$

Then Df is as before, $\det Df = r \in (\frac{1}{2}, 1)$, so f is a local diffeomorphism everywhere on U. Its image is the open annulus

$$V = \{(x, y) \in \mathbb{R}^2 \mid \tfrac{1}{4} < x^2 + y^2 < 1\} \setminus \{(x, 0) : x > 0\},$$

where the ray $\{(x, 0) : x > 0\}$ is removed to keep the angular coordinate single-valued. On U we have a global diffeomorphism $f : U \xrightarrow{\sim} V$.

Immersions: Curves and Surfaces

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Definition (Immersion).

Let $m, n \in \mathbb{N}$, $U \subseteq \mathbb{R}^m$ open, $V \subseteq \mathbb{R}^n$, and $f \in C^\infty(U, V)$. We say f is an *immersion* if for all $x \in U$, the Jacobian $D_x f \in M_{n \times m}(\mathbb{R})$ has *rank* m (its m columns are linearly independent).

Specific case.

- $m = 1$ (parametrized curve): $f(t) = (f_1(t), \ldots, f_n(t))$. Then

$$D_t f = \begin{pmatrix} f_1'(t) \\ \vdots \\ f_n'(t) \end{pmatrix}, \qquad f \text{ immersion } \Leftrightarrow D_t f \neq 0 \text{ for all } t.$$

Example

- $m = 2$, $n = 3$ (parametrized surface):
  $f(x_1, x_2) = (f_1(x_1, x_2), f_2(x_1, x_2), f_3(x_1, x_2))$ with

$$D_{(x_1,x_2)} f = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} \end{pmatrix} = \left(\frac{\partial f}{\partial x_1} \quad \frac{\partial f}{\partial x_2} \right),$$

so rank $Df = 2 \Leftrightarrow \frac{\partial f}{\partial x_1}$ and $\frac{\partial f}{\partial x_2}$ are linearly independent in $\mathbb{R}^3$. Equivalently, all 2×2 minors

$$b_1 = \det \begin{pmatrix} \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} \end{pmatrix}, b_2 = \det \begin{pmatrix} \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} \\ \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \end{pmatrix}, b_3 = \det \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}$$

do not vanish simultaneously (equivalently $\frac{\partial f}{\partial x_1} \times \frac{\partial f}{\partial x_2} \neq 0$).

Spherical Coordinates

Immersion of S^2

Parametrization. Let $(\theta, \varphi) \in (0, \pi) \times (0, 2\pi)$ and define

$$f(\theta, \varphi) = \big(\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta\big).$$

Then

$$\partial f/(\partial\theta) = (\cos\theta\cos\varphi,\ \cos\theta\sin\varphi,\ -\sin\theta)^T,$$
$$\partial f/(\partial\varphi) = (-\sin\theta\sin\varphi,\ \sin\theta\cos\varphi,\ 0)^T.$$

Rank condition. The 3×2 Jacobian $[\,\partial f/(\partial\theta)\ \ \partial f/(\partial\varphi)\,]$ has rank 2 whenever $\sin\theta \neq 0$, i.e. for $\theta \in (0, \pi)$, hence f is an immersion on that open strip.
Singularities (poles). At $\theta = 0$ (north pole) or $\theta = \pi$ (south pole), $\sin\theta = 0$ and thus $\partial f/(\partial\varphi) = 0$, so rank $Df \leq 1$; these are *singular points* for this chart.

Spherical Coordinates

Geometry.

- Fixing φ traces a *meridian* (e.g. the Greenwich meridian at $\varphi = 0$).
- Fixing θ traces a *parallel* (latitude circle); $\theta = \pi/2$ is the equator.

Charts. The sphere needs at least two charts to avoid the pole singularities; e.g. one chart missing the north pole and one missing the south pole.

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Summary

- Defined open and closed balls, spheres, and n–bricks (open, closed, boundary).
- Distinguished between interior, boundary, and exterior points of subsets $M \subseteq \mathbb{R}^n$.
- Defined open and closed sets, interiors, closures, and complements.
- Introduced relative (subspace) topology: sets open or closed in M via intersections with open/closed subsets of $\mathbb{R}^n$.
- Illustrated with visual examples in $\mathbb{R}^1$, $\mathbb{R}^2$, and $\mathbb{R}^3$ (segments, disks, cubes, spheres).

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Summary

- Smooth maps on open sets have derivatives of all orders; on non-open sets they are defined by smooth extension to an open neighborhood.
- The Jacobian $D_x f$ captures the best linear approximation; in polar coordinates det $Df = r$.
- Inverse Function Theorem: if det $D_{x_0} f \neq 0$, then f is locally a diffeomorphism; polar map on an annulus is a model example.
- Immersion $U \to \mathbb{R}^n$: full column rank Jacobian; curves require nonzero velocity; surfaces require f_u, f_v independent ($f_u \times f_v \neq 0$).
- Spherical chart is immersive off the poles; poles are chart singularities necessitating multiple charts to cover S^2.

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Thanks