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Combinatorial Mesh Calculus (CMC): Lecture 8

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Exterior Algebra

Graded-commutative rule.

If $A^\bullet = \bigoplus_{p \geq 0} A^p$ is a graded algebra and $v \in A^p$, $w \in A^q$, then

$$v\,w = (-1)^{pq}\,w\,v.$$

Definition (Exterior algebra).

Let R be a CRWU and V an R–module. The *exterior algebra* $\Lambda^\bullet V$ is the smallest *alternating* (associative, unital) R–algebra generated by V, i.e. the quotient of the tensor algebra $T(V)$ by the ideal generated by all $v \otimes v$ $(v \in V)$. It is graded

$$\Lambda^\bullet V = \bigoplus_{p=0}^{\infty} \Lambda^p V, \qquad \alpha \in \Lambda^p V,\ \beta \in \Lambda^q V \ \Rightarrow\ \alpha \wedge \beta \in \Lambda^{p+q} V,$$

Exterior Algebra

Definition (Exterior algebra).

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with $\Lambda^0 V = R$, $\Lambda^1 V = V$, and

$$\Lambda^2 V = \left\{ \sum_{k=1}^{N} \lambda_k\, v_k \wedge w_k \;\middle|\; N \in \mathbb{N},\ \lambda_k \in R,\ v_k, w_k \in V \right\}.$$

The product is denoted by $\wedge$ and satisfies graded-commutativity: if α is homogeneous of degree p and β of degree q, then $\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha$.

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Basic anti-commutation with a vector

Proposition.

Let R be a CRWU, V an R–module, $v \in V = \Lambda^1 V$ and $\alpha \in \Lambda^\bullet V$ homogeneous of degree p. Then

$$v \wedge \alpha = (-1)^p \, \alpha \wedge v.$$

In particular $v \wedge v = 0$ and, if p is odd, $v \wedge \alpha + \alpha \wedge v = 0$.

Proof.

In $\Lambda^\bullet V$ we have the graded-commutative law with degrees $\deg v = 1$ and $\deg \alpha = p$, hence $v \wedge \alpha = (-1)^{1 \cdot p} \alpha \wedge v$. Taking $p = 1$ yields $v \wedge v = -v \wedge v$, so $2(v \wedge v) = 0$. Since $v \wedge v$ is the image of $v \otimes v$ in the quotient $T(V)/\langle v \otimes v \rangle$, we have $v \wedge v = 0$ by construction (no characteristic assumption needed). If p is odd, $(-1)^p = -1$ and $v \wedge \alpha + \alpha \wedge v = 0$. $\square$

Dimension bounds and spanning sets

Corollary (Finite rank n).

Let R be a CRWU and V a free R–module of rank n with basis $e_1, \ldots, e_n$. Then:

1. If $p > n$, then $\Lambda^p V = 0$.
2. If $1 \leq p \leq n$, then the p–vectors

$$\{\, e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_p} \mid 1 \leq i_1 < \cdots < i_p \leq n \,\}$$

   span $\Lambda^p V$.

Proof

Proof.

(1) Any pure p–tensor $v_1 \wedge \cdots \wedge v_p$ is alternating and multilinear in the v_k. Express

$$v_k = \sum_{j=1}^{n} \lambda_{kj} e_j$$

and expand. Every term containing a repeated basis vector e_j vanishes because $e_j \wedge e_j = 0$. With only n distinct basis vectors available, no nonzero terms remain when $p > n$.

(2) By multilinearity, each $v_1 \wedge \cdots \wedge v_p$ is an R–linear combination of wedges of the basis vectors. Using $e_i \wedge e_j = -e_j \wedge e_i$ we can sort factors to increasing indices; terms with repeats vanish. Hence the displayed set spans $\Lambda^p V$. $\square$

Example in $\mathbb{R}^2$: wedge and determinants

Let $V = \mathbb{R}^2$ with basis $e_1 = (1, 0)$, $e_2 = (0, 1)$. Take

$$u = \lambda_1 e_1 + \lambda_2 e_2, \qquad v = \mu_1 e_1 + \mu_2 e_2, \qquad w = \kappa_1 e_1 + \kappa_2 e_2.$$

Compute $u \wedge v$. Using bilinearity and $e_1 \wedge e_1 = e_2 \wedge e_2 = 0$, $e_2 \wedge e_1 = -\, e_1 \wedge e_2$,

$$\begin{aligned} u \wedge v &= (\lambda_1 e_1 + \lambda_2 e_2) \wedge (\mu_1 e_1 + \mu_2 e_2) \\ &= \lambda_1 \mu_2 \,(e_1 \wedge e_2) + \lambda_2 \mu_1 \,(e_2 \wedge e_1) \\ &= (\lambda_1 \mu_2 - \lambda_2 \mu_1)\, e_1 \wedge e_2 = \det \begin{pmatrix} \lambda_1 & \lambda_2 \\ \mu_1 & \mu_2 \end{pmatrix} e_1 \wedge e_2. \end{aligned}$$

Compute $u \wedge v \wedge w$. Since $\Lambda^3 \mathbb{R}^2 = 0$ (corollary with $p = 3 > 2$),

$$u \wedge v \wedge w = 0.$$

The scalar coefficient of $e_1 \wedge e_2$ is the signed area (determinant) of the 2×2 matrix with rows (or columns) u, v; the third wedge vanishes because at most two independent directions live in $\mathbb{R}^2$.

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Example in $\mathbb{R}^3$: cross and triple products

Let $V = \mathbb{R}^3$ with basis e_1, e_2, e_3. Write

$$u = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3,$$
$$v = \mu_1 e_1 + \mu_2 e_2 + \mu_3 e_3,$$
$$w = \kappa_1 e_1 + \kappa_2 e_2 + \kappa_3 e_3.$$

$u \wedge v$. Expand and collect on the basis $\{e_1 \wedge e_2,\ e_2 \wedge e_3,\ e_3 \wedge e_1\}$:

$$\begin{aligned} u \wedge v =& (\lambda_1\mu_2 - \lambda_2\mu_1)\, e_1 \wedge e_2 + (\lambda_2\mu_3 - \lambda_3\mu_2)\, e_2 \wedge e_3 \\ &+ (\lambda_3\mu_1 - \lambda_1\mu_3)\, e_3 \wedge e_1. \end{aligned}$$

Identification with the cross product. Using the canonical isomorphism $* : \Lambda^2\mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$*(e_2 \wedge e_3) = e_1, \quad *(e_3 \wedge e_1) = e_2, \quad *(e_1 \wedge e_2) = e_3,$$

Note: We will discuss this *canonical isomorphism* $(*)$ in coming lectures.

Example in $\mathbb{R}^3$: cross and triple products

We get

$$*(u \wedge v) = \begin{pmatrix} \lambda_2\mu_3 - \lambda_3\mu_2 \\ \lambda_3\mu_1 - \lambda_1\mu_3 \\ \lambda_1\mu_2 - \lambda_2\mu_1 \end{pmatrix} = u \times v.$$

$u \wedge v \wedge w$. Expanding on the basis $e_1 \wedge e_2 \wedge e_3$ yields

$$u \wedge v \wedge w = \det \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \\ \kappa_1 & \kappa_2 & \kappa_3 \end{pmatrix} e_1 \wedge e_2 \wedge e_3.$$

The scalar is the scalar triple product $(u \times v) \cdot w$; it gives the oriented volume of the parallelepiped spanned by (u, v, w).

Bases of $\Lambda^\bullet\mathbb{R}^2$ and $\Lambda^\bullet\mathbb{R}^3$

For $V = \mathbb{R}^2$ with basis e_1, e_2:

$$\begin{aligned}\Lambda^\bullet V =& \Lambda^0 V \oplus \Lambda^1 V \oplus \Lambda^2 V\\ =& \text{Span}\{1\} \oplus \text{Span}\{e_1, e_2\} \oplus \text{Span}\{e_1 \wedge e_2\}.\end{aligned}$$

For $V = \mathbb{R}^3$ with basis e_1, e_2, e_3:

$$\begin{aligned}\Lambda^\bullet V =& \Lambda^0 V \oplus \Lambda^1 V \oplus \Lambda^2 V \oplus \Lambda^3 V\\ =& \text{Span}\{1\} \oplus \text{Span}\{e_1, e_2, e_3\}\\ &\oplus \text{Span}\{e_1 \wedge e_2,\ e_2 \wedge e_3,\ e_1 \wedge e_3\}\\ &\oplus \text{Span}\{e_1 \wedge e_2 \wedge e_3\}.\end{aligned}$$

Basis of $\Lambda^p V$ and Binomial Count

Theorem.

Let R be a CRWU and V an n–dimensional free R–module with basis $e_1, \ldots, e_n$. For $p \in \{0, 1, \ldots, n\}$ the set

$$\mathcal{B}_p = \{ e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_p} \mid 1 \le i_1 < \cdots < i_p \le n \}$$

is a basis of $\Lambda^p V$. In particular,

$$\dim \Lambda^p V = \binom{n}{p}.$$

Basis of $\Lambda^p V$ and Binomial Count

Proof.

Spanning: Any $v_1 \wedge \cdots \wedge v_p$ expands R–linearly in the basis $\{e_j\}$; terms with repeated indices vanish ($e_i \wedge e_i = 0$), and the graded–commutativity lets us sort indices increasingly, giving a combination of elements of $\mathcal{B}_p$. *Linear independence:* Consider the coordinate map $\phi : \Lambda^p V \to R^{\binom{n}{p}}$ that records coefficients w.r.t. $\mathcal{B}_p$; by construction its kernel is 0, so the family is independent. $\square$

Binomial theorem and dimensions. For t an indeterminate,

$$\sum_{p=0}^{n} \dim \Lambda^p V \; t^p = \sum_{p=0}^{n} \binom{n}{p} t^p = (1+t)^n.$$

At $t = 1$ this gives $\sum_{p=0}^{n} \binom{n}{p} = 2^n = \dim \Lambda^\bullet V$.

Exterior Power of a Linear Map

Definition.

Let R be a CRWU, V, W R–modules, $p \in \mathbb{N}$, and $f \in \mathrm{Hom}_R(V, W)$. The *$p$–th exterior power*

$$\Lambda^p f : \mathrm{Hom}_R\left(\Lambda^p V, \Lambda^p W\right)$$

is the unique R–linear map determined on decomposables by

$$(\Lambda^p f)(v_1 \wedge \cdots \wedge v_p) = f(v_1) \wedge \cdots \wedge f(v_p).$$

Uniqueness follows from multilinearity and alternation.

Exterior Power of a Linear Map

Example ($V = W = \mathbb{R}^2$).

Let $e_1 = (1, 0)$, $e_2 = (0, 1)$ and write

$$f(e_1) = a_{11}e_1 + a_{21}e_2, \qquad f(e_2) = a_{12}e_1 + a_{22}e_2.$$

Then

$$\begin{aligned}(\Lambda^2 f)(e_1 \wedge e_2) &= f(e_1) \wedge f(e_2) \\ &= (a_{11}e_1 + a_{21}e_2) \wedge (a_{12}e_1 + a_{22}e_2) \\ &= (a_{11}a_{22} - a_{21}a_{12})\, e_1 \wedge e_2 = \det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} e_1 \wedge e_2.\end{aligned}$$

Determinant via Top Exterior Power

Definition (Endomorphism).

$\mathrm{End}_R(V) = \mathrm{Hom}_R(V, V)$ is the R–algebra of R–linear maps $V \to V$.

Top exterior power and determinant.

If $\dim V = n$, then $\dim \Lambda^n V = \binom{n}{n} = 1$, so $\Lambda^n f : \Lambda^n V \to \Lambda^n V$ is multiplication by a unique scalar $\lambda \in R$:

$$(\Lambda^n f)(\alpha) = \lambda\, \alpha, \qquad \alpha \in \Lambda^n V.$$

We define $\det(f) := \lambda$.

Compatibility with matrices. Let $e = (e_1, \ldots, e_n)$ be a basis and $A = (f)_e^e \in M_{n\times n}(R)$ the matrix of f. Then

$$\det(f) = \det(A).$$

Determinant via Top Exterior Power

Proof sketch with details on $e_1 \wedge \cdots \wedge e_n$.

Write $f(e_j) = \sum_i a_{ij} e_i$. Then

$$
\begin{aligned}
(\Lambda^n f)(e_1 \wedge \cdots \wedge e_n) &= \bigwedge_{j=1}^{n} \left(\sum_{i=1}^{n} a_{ij} e_i \right) \\
&= \sum_{\sigma \in S_n} (\operatorname{sgn} \sigma)\, a_{1\sigma(1)} \cdots a_{n\sigma(n)}\, e_1 \wedge \cdots \wedge e_n \\
&= (\det A)\, e_1 \wedge \cdots \wedge e_n,
\end{aligned}
$$

so $\Lambda^n f$ is multiplication by $\det A$. $\square$

Multiplicativity of Determinant

Theorem. Let R be a CRWU, $\dim V = n$, and $f, g \in \mathrm{End}_R(V)$. Then

$$\det(g \circ f) = \det(g) \cdot \det(f).$$

Proof.

Functoriality of exterior powers gives $\Lambda^n(g \circ f) = \Lambda^n g \circ \Lambda^n f$. Since $\Lambda^n V$ is 1–dimensional, there exist unique $\lambda, \mu \in R$ with

$$\Lambda^n f = \lambda\,\mathrm{id}, \qquad \Lambda^n g = \mu\,\mathrm{id}.$$

Hence

$$\Lambda^n(g \circ f) = (\Lambda^n g) \circ (\Lambda^n f) = (\mu\,\mathrm{id}) \circ (\lambda\,\mathrm{id}) = (\mu\lambda)\,\mathrm{id},$$

so $\det(g \circ f) = \mu\lambda = \det(g)\det(f)$. $\square$

Visualization

$$
\begin{tikzcd}
\Lambda^n V \arrow[r, "\Lambda^n f"] \arrow[rr, bend left, "\Lambda^n(g \circ f)"] & \Lambda^n V \arrow[r, "\Lambda^n g"] & \Lambda^n V
\end{tikzcd}
$$

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Interior Product (Contraction)

Definition.

Let R be a CRWU, V an R–module. For $p \geq 1$, the *interior product* (contraction)

$$i: \; V \times \Lambda^p V^* \longrightarrow \Lambda^{p-1} V^*, \qquad (v, \omega) \mapsto i_v \omega$$

is the unique R–bilinear map such that:

(1) $p = 1: \quad i_v w = w(v) \in R, \qquad (v \in V, \; w \in V^*);$

(2) Leibniz rule (degree -1 derivation):

$$i_v(\omega \wedge \eta) = i_v \omega \wedge \eta + (-1)^p \, \omega \wedge i_v \eta,$$

for $\omega \in \Lambda^p V^*, \; \eta \in \Lambda^q V^*.$

By R–bilinearity, extend to all p and sums.

Interior Product: Computations in $\mathbb{R}^2$

Let $V = \mathbb{R}^2$ with basis e_1, e_2 and dual basis e^1, e^2.

On 1–forms:

$$i_{e_1}(e^1) = 1, \quad i_{e_1}(e^2) = 0, \qquad i_{e_2}(e^1) = 0, \quad i_{e_2}(e^2) = 1.$$

On 2–forms: using $i_v(\omega \wedge \eta) = i_v\omega \wedge \eta - \omega \wedge i_v\eta$ for $\omega \in \Lambda^1$,

$$i_{e_1}(e^1 \wedge e^2) = i_{e_1}(e^1) \wedge e^2 - e^1 \wedge i_{e_1}(e^2) = 1 \cdot e^2 - e^1 \wedge 0 = e^2,$$
$$i_{e_2}(e^1 \wedge e^2) = i_{e_2}(e^1) \wedge e^2 - e^1 \wedge i_{e_2}(e^2) = 0 \cdot e^2 - e^1 \cdot 1 = -e^1.$$

For $v = ae_1 + be_2$,

$$i_v(e^1 \wedge e^2) = a\, e^2 - b\, e^1.$$

This is the Hodge–dual of v in dimension 2.

On functions (0–forms): $i_v : \Lambda^0 V^* = R \to 0$ is the zero map.

Algebraic Identities for the Interior Prod.

Proposition.

Let R be a CRWU and V an R–module.

1. For any $v \in V$, $i_v \circ i_v = 0$ on $\Lambda^\bullet V^*$.
2. The canonical map

$$\phi : \Lambda^p(V^*) \longrightarrow (\Lambda^p V)^*$$

defined on decomposables by

$$\phi(w_1 \wedge \cdots \wedge w_p)(v_1 \wedge \cdots \wedge v_p) = i_{v_p} \circ \cdots \circ i_{v_1}(w_1 \wedge \cdots \wedge w_p)$$

is an isomorphism when V is finite-dimensional. Equivalently,

$$\phi(w_1 \wedge \cdots \wedge w_p)(v_1 \wedge \cdots \wedge v_p) = \det(w_i(v_j))_{1 \le i,j \le p}\,.$$

Proof

Proof.

(1) It suffices to check on $\omega = w_1 \wedge \cdots \wedge w_p$ with $w_i \in V^*$. The operator i_v inserts v into the first slot of the alternating p–form (up to signs).
Inserting the same vector twice yields 0 by alternation, hence $i_v^2\omega = 0$; linearity gives the claim.
(2) For decomposables the two displayed formulas agree (straight computation using the Leibniz rule shows the cofactor expansion).
In a basis $\{e_i\}$ with dual $\{e^i\}$, both $\Lambda^p(V^*)$ and $(\Lambda^p V)^*$ have the same rank $\binom{n}{p}$, and ϕ sends the basis element $e^{i_1} \wedge \cdots \wedge e^{i_p}$ to the functional that picks the coefficient of $e_{i_1} \wedge \cdots \wedge e_{i_p}$; hence ϕ is bijective. $\square$

Worked Contraction Example

For $w_1, w_2 \in V^*$ and $v_1, v_2 \in V$,

$$
\begin{aligned}
i_{v_2} \circ i_{v_1}(w_1 \wedge w_2) &= i_{v_2}\left(i_{v_1}(w_1) \wedge w_2 - w_1 \wedge i_{v_1}(w_2)\right) \\
&= i_{v_2}(w_1(v_1))\, w_2 \; - \; i_{v_2}(w_1)\, w_2(v_1) \; - \; i_{v_2}(w_2(v_1))\, w_1 \\
&= 0 \cdot w_2 \; - \; w_1(v_2)\, w_2(v_1) \; - \; 0 \cdot w_1 \; + \; w_2(v_2)\, w_1(v_1) \\
&= \begin{vmatrix} w_1(v_1) & w_1(v_2) \\ w_2(v_1) & w_2(v_2) \end{vmatrix}.
\end{aligned}
$$

Thus, for $p = 2$,

$$
\phi(w_1 \wedge w_2)(v_1 \wedge v_2) = \det \begin{pmatrix} w_1(v_1) & w_1(v_2) \\ w_2(v_1) & w_2(v_2) \end{pmatrix},
$$

which matches the determinant–via–top–exterior–power paradigm.

Summary of Lecture 07

Main Ideas: Exterior Algebra and Determinant Structures

- **Exterior algebra** $\Lambda^\bullet V$: smallest alternating, associative, unital algebra generated by a module V over a CRWU.
- It is a **graded algebra**:

$$\Lambda^\bullet V = \bigoplus_{p=0}^{n} \Lambda^p V, \quad \alpha \in \Lambda^p V,\ \beta \in \Lambda^q V \Rightarrow \alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha.$$

- **Basis:** For $\dim V = n$, $\{e_{i_1} \wedge \cdots \wedge e_{i_p} \mid 1 \le i_1 < \cdots < i_p \le n\}$ is a basis of $\Lambda^p V$.
- **Dimension:** $\dim \Lambda^p V = \binom{n}{p}$, and

$$\sum_{p=0}^{n} \dim \Lambda^p V \ t^p = (1+t)^n.$$

Summary of Lecture 07

Main Ideas: Exterior Algebra and Determinant Structures

- **Exterior power of maps:**
  $\Lambda^p f(v_1 \wedge \cdots \wedge v_p) = f(v_1) \wedge \cdots \wedge f(v_p)$.
- **Determinant:** For $f \in \mathrm{End}(V)$, $\Lambda^n f$ acts on the one-dimensional $\Lambda^n V$ by multiplication with $\det(f)$, and $\det(g \circ f) = \det(g)\det(f)$.
- **Interior product:** $i_v : \Lambda^p V^* \to \Lambda^{p-1} V^*$ defined by

$$i_v(\omega \wedge \eta) = i_v\omega \wedge \eta + (-1)^p \omega \wedge i_v\eta, \quad i_v w = w(v).$$

- **Canonical isomorphism:** $\Lambda^p(V^*) \cong (\Lambda^p V)^*$, with

$$\phi(w_1 \wedge \cdots \wedge w_p)(v_1 \wedge \cdots \wedge v_p) = \det\big(w_i(v_j)\big)_{1 \le i,j \le p}.$$

Roadmap

Set
X

$\downarrow$

Group-like structure
Binary op. $*: R \times R \to R$

$\downarrow$

Ring-like structure
$(R, +, \cdot)$ with $+, \cdot : R \times R \to R$

$\downarrow$

Module
$(V, +, \cdot)$ with $+: V \times V \to V$, $\cdot : R \times V \to V$

$\downarrow$

Algebra
$(A, +, \cdot, *)$ combining ring & module over R

Tensor Algebra
$T(V) = \bigoplus_{i \geq 0} T^i(V)$

Exterior Algebra
$\Lambda(V) = T(V)/\langle v \otimes v \rangle$

Symmetric Algebra
$S(V) = T(V)/\langle v \otimes w - w \otimes v \rangle$

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Thanks