Combinatorial Mesh Calculus (CMC): Lecture 7

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Definition

by Universal Property

Let R be a **CRWU**. Let U, V be R–modules. A tensor product of U and V is an R–module X together with a bilinear map

$$\tau : U \times V \longrightarrow X$$

such that for every R–module W, *precomposition by* τ yields a natural isomorphism

$$\Phi_W : \mathrm{Hom}_R(X, W) \simeq \mathcal{L}(U, V; W), \qquad \Phi_W(\alpha) = \alpha \circ \tau.$$

Equivalently: bilinear maps $U \times V \to W$ *factor uniquely* through a **linear** map $X \to W$. We denote this essentially unique object by $U \otimes_R V$, and write $u \otimes v := \tau(u, v)$.

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Existence, Uniqueness, and Identities

Theorem (Existence and uniqueness up to unique iso)

For R a CRWU and U, V R–modules, a tensor product $U \otimes_R V$ exists; any two such pairs (X, τ) and (X', τ') are uniquely isomorphic by the universal property.

Canonical identities in $U \otimes_R V$ (forced by bilinearity)

$$(u_1 + u_2) \otimes v = u_1 \otimes v + u_2 \otimes v, \qquad u \otimes (v_1 + v_2) = u \otimes v_1 + u \otimes v_2,$$

$$(\lambda u) \otimes v = u \otimes (\lambda v) = \lambda (u \otimes v), \quad \forall\, u, u_1, u_2 \in U,\ v, v_1, v_2 \in V,\ \lambda \in R.$$

Identities

Finite bases.

If $e = (e_1, \dots, e_m)$ is a basis of U and $f = (f_1, \dots, f_n)$ a basis of V, then

$$\mathcal{B} = \{\, e_i \otimes f_j \mid 1 \leq i \leq m,\ 1 \leq j \leq n \,\}$$

is a basis of $U \otimes_R V$, hence

$$\dim(U \otimes_R V) = (\dim U)(\dim V) = mn.$$

Explicit Basis Example

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Example.

Over $R = \mathbb{R}$, let $U = \mathbb{R}^2$ with basis

$$\begin{aligned} e_1 &= (1,0), \\ e_2 &= (0,1) \end{aligned}$$

and $V = \mathbb{R}^3$ with basis

$$\begin{aligned} f_1 &= (1,0,0), \\ f_2 &= (0,1,0), \\ f_3 &= (0,0,1). \end{aligned}$$

Then a basis of $\mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^3$ is

$$\{\, e_1 \otimes f_1,\ e_1 \otimes f_2,\ e_1 \otimes f_3,\ e_2 \otimes f_1,\ e_2 \otimes f_2,\ e_2 \otimes f_3 \,\},$$

so $\dim(\mathbb{R}^2 \otimes \mathbb{R}^3) = 6 = 2 \cdot 3$.

Symmetry for $\otimes$

Proposition (Symmetry)

There is a canonical R–module isomorphism

$$\sigma_{U,V} : U \otimes_R V \simeq V \otimes_R U, \qquad \sigma_{U,V}(u \otimes v) = v \otimes u,$$

characterized by bilinearity; it is natural in both variables.

Proof.

The map $(u, v) \mapsto v \otimes u$ is bilinear $U \times V \to V \otimes U$, hence by universality induces a unique linear map $U \otimes V \to V \otimes U$. Its inverse is the same recipe with U, V swapped. $\square$

Unit for $\otimes$

Proposition (Unit object)

There are canonical isomorphisms

$$\lambda_U : R \otimes_R U \simeq U, \qquad \lambda_U(1 \otimes u) = u,$$

$$\rho_U : U \otimes_R R \simeq U, \qquad \rho_U(u \otimes 1) = u,$$

extended R–linearly.

Proof.

Bilinearity of $(\lambda, u) \mapsto \lambda u$ (resp. $(u, \lambda) \mapsto \lambda u$) gives unique linear maps inverse to $u \mapsto 1 \otimes u$ (resp. $u \mapsto u \otimes 1$). $\square$

Definition and Isomorphism Criteria

Definition

Let U, V be R–modules. Define the R–linear map

$$\rho : U^* \otimes_R V \longrightarrow \mathrm{Hom}_R(U, V), \qquad \rho(f \otimes v)(u) := f(u)\, v.$$

Proposition

If $U = R$ or $V = R$, or if U and V are finite-dimensional, then ρ is an isomorphism.

Proposition (Proof)

Proof.

(1) If $U = R$, then $U^* \cong R$ via $f \mapsto f(1)$; $\rho(\lambda \otimes v)(\mu) = \lambda\mu\, v$, identifying $R \otimes V \cong V$ and $\mathrm{Hom}_R(R, V) \cong V$, so ρ is the identity under these identifications.

(2) If $V = R$, then $\rho : U^* \otimes R \to \mathrm{Hom}_R(U, R) = U^*$ identifies with the unit isomorphism $U^* \otimes R \cong U^*$.

(3) If U, V are finite-dimensional with bases $(e_i)_{i=1}^m$ and $(v_j)_{j=1}^n$ and dual (e^i), then

$$U^* \otimes V \cong R^m \otimes R^n \cong R^{mn} \cong \mathrm{Hom}_R(U, V),$$

and $\rho(e^i \otimes v_j)$ is the rank–one map $u \mapsto e^i(u)\, v_j$. These mn maps form the standard basis of $\mathrm{Hom}_R(U, V)$ with respect to (e_i) and (v_j). Hence ρ is bijective. □

A Structural Corollary

Corollary (Polarization into V and V^*)

Let R be a CRWU and V a finite-dimensional R–module. Any R–module obtained from V and R by iterating constructions using tensor products and Hom_R (in particularly taking duals, i.e., $\mathrm{Hom}_R(\cdot, R)$) is canonically isomorphic to

$$\underbrace{V \otimes \cdots \otimes V}_{m \text{ times}} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{n \text{ times}} \quad \text{for some } m, n \in \mathbb{N}.$$

Proof.

Proceed by structural induction on the set S of such expressions. The base objects V and R are in S. Now take $X = V^{\otimes m} \otimes (V^*)^{\otimes n} \in S$ and $Y = V^{\otimes p} \otimes (V^*)^{\otimes q} \in S$ for some $m, n, p, q \in \mathbb{N}$. Then $X \otimes Y \simeq V^{\otimes (m+p)} \otimes (V^*)^{\otimes (n+q)} \in S$ and $\mathrm{Hom}_R(X, Y) \simeq X^* \otimes Y \simeq V^{\otimes (n+p)} \otimes (V^*)^{\otimes (m+q)} \in S$. $\square$

Dual of a Tensor Product

Note

If U, V are finite-dimensional R–modules, then there is a canonical isomorphism

$$\begin{aligned}(U \otimes_R V)^* &\cong \mathrm{Hom}_R(U \otimes V, R) \\ &\cong \mathrm{Hom}_R\big(U, \mathrm{Hom}_R(V, R)\big) \cong U^* \otimes_R V^*.\end{aligned}$$

Proof.

Use the currying isomorphism $\mathrm{Hom}_R(U \otimes V, R) \cong \mathcal{L}(U, V; R)$ and then apply $\rho : U^* \otimes V^* \xrightarrow{\sim} \mathcal{L}(U, V; R)$ (finite-dimensional case). Naturality shows canonicity. $\square$

Definition and Basic Properties

Definition (Direct sum of two R–modules).

Let R be a CRWU and let U, V be R–modules. Define the *direct sum* $U \oplus V$ to be the cartesian product $U \times V$ endowed with componentwise addition and scalar multiplication:

$$(u_1 \oplus v_1) +_{U\oplus V} (u_2 \oplus v_2) = (u_1 +_U u_2) \oplus (v_1 +_V v_2),$$
$$\lambda \cdot_{U\oplus V} (u \oplus v) = (\lambda \cdot_U u) \oplus (\lambda \cdot_V v),$$

for all $u, u_1, u_2 \in U$, $v, v_1, v_2 \in V$, and $\lambda \in R$. We **write elements** of $U \oplus V$ as $u \oplus v$ (with $u \in U$, $v \in V$). The **zero** is $0_{U\oplus V} = 0_U \oplus 0_V$, and the **additive inverse** is

$$-(u \oplus v) = (-u) \oplus (-v).$$

Definition and Basic Properties

The **canonical injections** and **projections** are

$$\begin{aligned} i_U &: U \to U \oplus V, \quad i_U(u) = u \oplus 0, & i_V &: V \to U \oplus V, \quad i_V(v) = 0 \oplus v, \\ \pi_U &: U \oplus V \to U, \quad \pi_U(u \oplus v) = u, & \pi_V &: U \oplus V \to V, \quad \pi_V(u \oplus v) = v, \end{aligned}$$

satisfying $\pi_U \circ i_U = \mathrm{id}_U$, $\pi_V \circ i_V = \mathrm{id}_V$, and $\pi_U \circ i_V = \pi_V \circ i_U = 0$.
Universal property (biproduct): For any R–module W and maps $f : U \to W$, $g : V \to W$, there is a unique R–linear map

$$f \oplus g : U \oplus V \to W, \qquad (f \oplus g)(u \oplus v) = f(u) +_W g(v),$$

and dually, given $h : W \to U$, $k : W \to V$, there is a unique R–linear map

$$\langle h, k \rangle : W \to U \oplus V, \qquad \langle h, k \rangle(w) = h(w) \oplus k(w).$$

Basic Properties

Finite bases and dimension.

If U, V are finite-dimensional with bases $e = (e_1, \ldots, e_m)$ and $f = (f_1, \ldots, f_n)$, then

$$\{\, e_1 \oplus 0, \ldots, e_m \oplus 0,\ 0 \oplus f_1, \ldots, 0 \oplus f_n \,\}$$

is a basis of $U \oplus V$,
hence $\dim(U \oplus V) = \dim U + \dim V = m + n$.

Example

$\mathbb{R} \oplus \mathbb{R}^2 \simeq \mathbb{R}^3$

Let $R = \mathbb{R}$. Take $\mathbb{R}$ as a 1–dimensional real module with the standard basis $e_1 = 1$, and $\mathbb{R}^2$ with the standard basis $f_1 = (1, 0),\ f_2 = (0, 1)$. Then the direct sum $\mathbb{R} \oplus \mathbb{R}^2$ comes with the basis

$$\mathcal{B} = \{e_1 \oplus 0_{\mathbb{R}^2},\ 0 \oplus f_1,\ 0 \oplus f_2\}.$$

Expanded, $\mathcal{B}$ becomes

$$\mathcal{B} = \{1 \oplus (0, 0),\ 0 \oplus (1, 0),\ 0 \oplus (0, 1)\}.$$

We see that $\mathcal{B}$ is equivalent to the standard basis of $\mathbb{R}^3$:

$$\mathcal{B}' = \{(1, 0, 0),\ (0, 1, 0),\ (0, 0, 1)\}.$$

Canonical Isomorphisms

(1) Commutativity of $\oplus$.

$$\chi : U \oplus V \simeq V \oplus U, \quad \chi(u \oplus v) = v \oplus u.$$

χ is linear with inverse itself.

(2) Associativity of $\oplus$.

$$\alpha : (U \oplus V) \oplus W \simeq U \oplus (V \oplus W), \quad \alpha((u \oplus v) \oplus w) = u \oplus (v \oplus w).$$

Linear with inverse $u \oplus (v \oplus w) \mapsto (u \oplus v) \oplus w$.

Canonical Isomorphisms

(3) Unit for $\oplus$.

$$\eta : U \oplus 0 \simeq U, \quad \eta(u \oplus 0) = u, \qquad \zeta : 0 \oplus U \simeq U, \quad \zeta(0 \oplus u) = u.$$

Both linear with evident inverses.

(4) Dual of a direct sum.

$$\Delta : U^* \oplus V^* \simeq (U \oplus V)^*, \quad \Delta(f \oplus g)(u \oplus v) = f(u) + g(v).$$

Linear and bijective (construct inverse by restriction to the summands).

Canonical Isomorphisms

(5) Distributivity of $\otimes$ over $\oplus$ (left).

$$\delta : (U \oplus V) \otimes W \simeq (U \otimes W) \oplus (V \otimes W), \ \delta\big((u \oplus v) \otimes w\big) = (u \otimes w) \oplus (v \otimes w).$$

Well-defined by bilinearity; inverse given by
$(u \otimes w) \oplus (v \otimes w) \mapsto (u \oplus v) \otimes w$.

(6) Distributivity (right).

$$\delta' : U \otimes (V \oplus W) \simeq (U \otimes V) \oplus (U \otimes W), \ \delta'(u \otimes (v \oplus w)) = (u \otimes v) \oplus (u \otimes w).$$

Analogous proof.

External Direct Sums over an Index Set

Definition (External direct sum).

Let R be a CRWU, I a (possibly infinite) set, and $\{A_i\}_{i\in I}$ a family of R–modules. Define

$$\bigoplus_{i\in I} A_i := \{\, (a_i)_{i\in I} \mid a_i \in A_i, \text{ and } a_i = 0 \text{ for all but finitely many } i \,\}.$$

Operations are pointwise:

$$(a_i)_{i\in I} + (b_i)_{i\in I} = (a_i + b_i)_{i\in I}, \qquad \lambda(a_i)_{i\in I} = (\lambda a_i)_{i\in I}.$$

We write a typical element as a finite sum $a_{i_1} \oplus \cdots \oplus a_{i_n}$ with $a_{i_k} \in A_{i_k}$.

Example

Polynomials as an Infinite Direct Sum

For a CRWU R,

$$R[x] = \bigoplus_{i=0}^{\infty} \mathrm{Span}_R\{x^i\} = \{a_0 \mid a_0 \in R\} \;\oplus\; \{a_1 x \mid a_1 \in R\} \;\oplus\; \cdots\;,$$

since any polynomial has finitely many nonzero coefficients. The summand $\mathrm{Span}_R\{x^i\} \cong R$ records the x^i–coefficient.

R–Algebras

Definition (Algebra over a CRWU).

Let $(V, +)$ be an abelian group with scalar multiplication $\cdot : R \times V \to V$ making V an R–module, and a bilinear product $* : V \times V \to V$. Then $(V, +, *, \cdot)$ is an R–algebra if $*$ is distributive over $+$ and R–linear in each slot:

$$(x + y) * z = x * z + y * z, \qquad x * (y + z) = x * z + x * z,$$
$$(\lambda x) * y = \lambda(x * y), \qquad x * (\lambda y) = \lambda(x * y),$$

for all $x, y, z \in V$, $\lambda \in R$. If $*$ is associative/unital/commutative, we say the algebra has that property.

Remark:

Lie algebras use a different product (the Lie bracket) which is bilinear, alternating, and satisfies Jacobi; it is neither associative nor commutative.

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Examples

Polynomial Algebras

- $(R[x], +, \cdot)$ with usual polynomial multiplication is an associative, unital (unit 1), and commutative R–algebra.
- $R[x_1, \ldots, x_n]$ is likewise associative, unital, commutative, and infinite–dimensional (unless $n = 0$).

Tensor Algebra $T(V)$

Definition (Tensor algebra). Let R be a CRWU and V an R–module. The tensor algebra is

$$T(V) := \bigoplus_{i=0}^{\infty} T^i(V),\ T^0(V) := R,\ T^1(V) := V, \quad T^i(V) := V^{\otimes i}\ (i \geq 2),$$

with multiplication induced by tensor concatenation:

$$T^i(V) \times T^j(V) \to T^{i+j}(V), \qquad (x, y) \mapsto x \otimes y.$$

Thus $T(V)$ is the smallest associative unital R–algebra containing V. If dim $V = n < \infty$ with basis $e_1, \ldots, e_n$, then dim $T^i(V) = n^i$ and a basis is

$$\{ e_{j_1} \otimes \cdots \otimes e_{j_i} \mid 1 \leq j_k \leq n \}.$$

The full algebra has the graded decomposition $T(V) = \bigoplus_{i=0}^{\infty} T^i(V)$.

Degree and Graded Algebras

Degree. If $x \in T^i(V)$ and $y \in T^j(V)$, then

$$x \otimes y \in T^{i+j}(V), \qquad \deg(x) = i.$$

Definition (Graded R–algebra). An R–algebra A is *graded* if $A = \bigoplus_{i=0}^{\infty} A_i$ as R–modules and

$$A_i \cdot A_j \subseteq A_{i+j} \qquad \text{for all } i, j \geq 0.$$

Examples. Polynomial algebras $R[x_1, \ldots, x_n]$ and tensor algebras $T(V)$ are graded by total degree.

Concrete Degree Computation in $T(\mathbb{R}^2)$

Let $V = \mathbb{R}^2$ with basis $e_1 = (1,0)$, $e_2 = (0,1)$. Consider

$$x = 2\, e_1 \otimes e_1 \;-\; 3\, e_1 \otimes e_2 \in T^2(V),$$
$$y = \; e_1 \otimes e_2 \otimes e_1 \;+\; 2\, e_1 \otimes e_1 \otimes e_1 \in T^3(V).$$

Then $\deg(x) = 2$, $\deg(y) = 3$, and

$$x \otimes y \in T^5(V), \qquad y \otimes x \in T^5(V).$$

Each is a linear combination of basis monomials of length 5 in $\{e_1, e_2\}$.

A Ring with $r^2 = 0$ is Anti-commutative

Proposition.

Let R be a ring such that $r^2 = 0$ for all $r \in R$. Then for all $r, s \in R$,

$$rs + sr = 0 \quad \text{(i.e. } rs = -sr\text{).}$$

Proof.

Compute $(r+s)^2$ in two ways. On the one hand, by hypothesis $(r+s)^2 = 0$. On the other hand,

$$(r+s)^2 = r^2 + rs + sr + s^2 = 0 + rs + sr + 0 = rs + sr.$$

Hence $rs + sr = 0$ for all $r, s \in R$. $\square$

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Alternating (Exterior-type) Algebras

Definition (Alternating algebra).

Let R be a CRWU and $A = \bigoplus_{i=0}^{\infty} A_i$ a graded associative unital R–algebra with $1 \in A_0$. We say A is *alternating* if

$$v^2 = 0 \quad \text{for all } v \in A_1.$$

Equivalently (over $2 \in R^\times$), for $v, w \in A_1$,

$$vw = -wv.$$

Heuristic: Elements of degree 1 anticommute; the algebra is generated in degree 1 subject to these relations.

Key Identities in A-G-A

Proposition.

Let $A = \bigoplus_{i=0}^{\infty} A_i$ be an alternating graded R–algebra (associative, unital). Then:

1. If $j \in \mathbb{N}$ is odd and $v \in A_j$ is homogeneous, then $v^2 = 0$.
2. If $v \in A_i$ and $w \in A_j$ are homogeneous, then

$$v\,w = (-1)^{ij}\,w\,v.$$

Proof

(2) First prove the claim for $v, w \in A_1$ (this is the defining property: $vw = -wv$). Extend to arbitrary homogeneous $v \in A_i$ and $w \in A_j$ by writing

Alternating Graded Algebras

Proof.

$$v = v_1 \cdots v_i, \qquad w = w_1 \cdots w_j \qquad (v_k, w_\ell \in A_1),$$

and moving each v_k past all w_ℓ using $v_k w_\ell = -w_\ell v_k$. This introduces ij sign changes:

$$v\,w = (v_1 \cdots v_i)(w_1 \cdots w_j) = (-1)^{ij}(w_1 \cdots w_j)(v_1 \cdots v_i) = (-1)^{ij} w\,v.$$

(1) Put $w = v$ with $\deg v = j$; then

$$v^2 = (-1)^{jj} v^2 = (-1)^{j^2} v^2.$$

If j is odd, $(-1)^{j^2} = -1$, hence $v^2 = -v^2$, so $2v^2 = 0$. Over any CRWU in which the alternating relation is imposed (e.g. exterior algebras over $\mathbb{Z}$ or fields of char $\neq 2$), this forces $v^2 = 0$. In particular, in the exterior algebra $\bigwedge V$, $v \wedge v = 0$ for all odd-degree homogeneous v. $\square$

Worked Sign Example

Let $v = v_1 v_2$ with $v_1, v_2 \in A_1$ (so $\deg v = 2$) and $w = w_1 w_2 w_3$ with $w_k \in A_1$ (so $\deg w = 3$). Then by the previous proposition,

$$wv = (-1)^{(\deg w)(\deg v)}\, vw = (-1)^{3\cdot 2}\, vw = (+1)\, vw.$$

Concretely, moving v_1 past w_1, w_2, w_3 produces 3 sign flips, and moving v_2 past w_1, w_2, w_3 produces another 3 sign flips, totaling 6 flips: an even number $\Rightarrow$ no net sign.

Summary

- $U \otimes_R V$ represents bilinear maps: $\mathrm{Hom}_R(U \otimes V, W) \cong \mathcal{L}(U, V; W)$; basis tensors $e_i \otimes f_j$ give $\dim(U \otimes V) = \dim U \cdot \dim V$.
- Canonical isomorphisms: $U \otimes V \cong V \otimes U$, $R \otimes U \cong U$, and $\rho : U^* \otimes V \to \mathrm{Hom}_R(U, V)$ (iso in finite-dimensional cases).
- Direct sums: $U \oplus V$ has block basis and $\dim(U \oplus V) = \dim U + \dim V$; $(U \oplus V) \otimes W \cong (U \otimes W) \oplus (V \otimes W)$; $(U \oplus V)^* \cong U^* \oplus V^*$.
- R–algebras: R–modules with a bilinear product; polynomial algebras are associative, unital, commutative.
- Tensor algebra $T(V) = \bigoplus_{i \geq 0} V^{\otimes i}$ is graded; $\dim T^i(V) = (\dim V)^i$.
- Alternating graded algebras impose $v^2 = 0$ for $v \in A_1$; consequence: graded sign rule $vw = (-1)^{ij} wv$, and odd-degree squares vanish.

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Thanks