Combinatorial Mesh Calculus (CMC): Lecture 4

Lectured by: Dr. Kiprian Berbatov[1]

Lecture Notes Compiled by: Muhammad Azeem[1]

Under the supervision of: Prof. Andrey P. Jivkov[1]

[1] Department of Mechanical and Aerospace Engineering, The University of Manchester, Oxford Road, Manchester M13 9PL, UK

Module over a Commutative Ring with Unity (CRWU)

Definition.

Let R be a commutative ring with unity (CRWU), and let $(V, +, -, 0)$ be an abelian (commutative) group under addition. A *scalar multiplication* $* : R \times V \to V$ makes V an R–**module** if the following hold for all $\lambda, \mu \in R$ and $v, w \in V$:

$(\lambda\mu) * v = \lambda * (\mu * v)$	(Associativity of scalar mult.)
$(\lambda + \mu) * v = \lambda * v + \mu * v$	(Distributivity over ring addition)
$\lambda * (v + w) = \lambda * v + \lambda * w$	(Distributivity over module addition)
$1_R * v = v$	(Unital condition)

Notation: $(V, +, *, 0)$ is called an R–*module*. If R is a field, V is a **vector space over** R.

Interpretation and Diagram

Idea

Modules generalize vector spaces—where scalars come not from a field, but from a ring.

$$\begin{tikzcd} R \arrow[dr] & & V \arrow[dl] \\ & R \times V \arrow[dr, "*"] & \\ & & V \end{tikzcd}$$

$$(\lambda, v) \mapsto \lambda * v$$

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Examples of R–Modules

Let R be a CRWU, $m, n \in \mathbb{N}$.

1. Coordinate Module:

$$R^n = \{(a_1, \ldots, a_n) \mid a_i \in R\}.$$

Addition and scalar multiplication defined by

$$(a_1, \ldots, a_n) + (b_1, \ldots, b_n) = (a_1 + b_1, \ldots, a_n + b_n),$$

$$\lambda * (a_1, \ldots, a_n) = (\lambda a_1, \ldots, \lambda a_n).$$

Then $(R^n, +, *)$ is an R–module. When R is a field, this is a vector space.

2. Matrix Module:

$$M_{m\times n}(R) = \{[a_{ij}]_{m\times n} \mid a_{ij} \in R\},$$

with addition $(A + B)_{ij} = a_{ij} + b_{ij}$, and scalar multiplication $(\lambda A)_{ij} = \lambda a_{ij}$.

Examples (continued)

3. Zero Module: $\{0\}$ with $r * 0 = 0$ for all $r \in R$.

4. Free Module: Let $S = \{e_1, \ldots, e_n\}$ be a finite set. Define

$$\mathrm{Free}_R(S) = \left\{ \sum_{i=1}^{n} \lambda_i e_i \mid \lambda_i \in R \right\}.$$

Addition and scalar multiplication:

$$(\lambda_1 e_1 + \cdots + \lambda_n e_n) + (\mu_1 e_1 + \cdots + \mu_n e_n) = (\lambda_1 + \mu_1)e_1 + \cdots + (\lambda_n + \mu$$

$$r(\lambda_1 e_1 + \cdots + \lambda_n e_n) = (r\lambda_1)e_1 + \cdots + (r\lambda_n)e_n.$$

Then $\mathrm{Free}_R(S)$ is an R–module.

Free Module on Infinite Basis

If S is infinite, $\text{Free}_R(S)$ consists of **finite linear combinations**:

$$\lambda_1 e_{i_1} + \cdots + \lambda_k e_{i_k}, \quad \lambda_i \in R.$$

Example: $S = \{1, x, x^2, x^3, \ldots\}$ then

$$\text{Free}_R(S) = R[x],$$

the ring of polynomials over R—a free R–module with basis

$$\{1, x, x^2, \ldots\}$$

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Linear Independence and Span

Definition: Let V be an R–module, $e_1, \ldots, e_n \in V$.

1. **Linear Independence:** $\{e_1, \ldots, e_n\}$ is linearly independent if

$$\lambda_1 e_1 + \cdots + \lambda_n e_n = 0 \Rightarrow \lambda_1 = \cdots = \lambda_n = 0.$$

2. **Linear Dependence:** The set is linearly dependent if there exist coefficients, not all zero, satisfying

$$\lambda_1 e_1 + \cdots + \lambda_n e_n = 0.$$

3. **Span:**

$$\mathrm{Span}_R\{e_1, \ldots, e_n\} = \left\{ \sum_{i=1}^{n} \lambda_i e_i \mid \lambda_i \in R \right\}.$$

4. **Basis:** A set $\{e_1, \ldots, e_n\}$ is a basis of V if it is linearly independent and spans V.

Linear independence over $\mathbb{Z}$ and $\mathbb{Q}$

Example

Let R be $\mathbb{Q}$ or $\mathbb{Z}$, $V = R^2$, $v_1 = (2,0)$, $v_2 = (3,0)$. Then the vectors v_1 and v_2 are linearly dependent since $3v_1 - 2v_2 = 0$. When $R = \mathbb{Q}$, we can find reciprocals, and so $v_2 = (3/2)v_1$. However, when $R = \mathbb{Z}$, we cannot express one of the vectors as a multiple of the other.

Remark

If R is a field and $\mu_1 v_1 + \cdots + \mu_n v_n = 0$ with $\mu_i \neq 0$, then we can divide:

$$v_i = \sum_{j \neq i} \frac{-\mu_j}{\mu_i} v_j.$$

However, for general modules (where R is not a field), division may not be possible, so such reduction cannot be performed.

Standard Basis of R^n

Example:

For R^n define

$$e_1 = (1, 0, \ldots, 0), \; e_2 = (0, 1, 0, \ldots, 0), \; \ldots, \; e_n = (0, \ldots, 0, 1).$$

Claim: $\{e_1, \ldots, e_n\}$ is a basis of R^n.

Proof:

- Every $v = (a_1, \ldots, a_n) \in R^n$ can be written uniquely as

$$v = a_1 e_1 + \cdots + a_n e_n.$$

- If $\sum \lambda_i e_i = 0$, then all $\lambda_i = 0$. Hence independence.

Therefore $\{e_i\}$ is a basis.

Matrix Unit Basis of $M_{m\times n}(R)$

Example:

For each (i,j) define $E_{ij} \in M_{m\times n}(R)$ by

$$(E_{ij})_{kl} = \begin{cases} 1, & k=i,\ l=j, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\{E_{ij} \mid 1 \le i \le m, 1 \le j \le n\}$ is a basis of $M_{m\times n}(R)$.

Example (for $m = n = 2$):

$$E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Any $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ can be written as

$$A = aE_{11} + bE_{12} + cE_{21} + dE_{22}.$$

Other Basis Examples

Zero Module:

ϕ is a basis of $\{0\}$.

Free Module:

If S is a set, S itself forms a basis of $\text{Free}_R(S)$.

Polynomial Module:

For $S = \{1, x, x^2, \ldots\}$,

$$R[x] = \text{Free}_R(S)$$

with basis $\{1, x, x^2, \ldots\}$.

Uniqueness of Representation

Proposition.

Let R be a CRWU, V an R–module, and $S = \{e_1, \ldots, e_n\} \subseteq V$. Then S is a basis of V if and only if every $v \in V$ can be expressed uniquely as

$$v = \lambda_1 e_1 + \cdots + \lambda_n e_n, \quad \lambda_i \in R.$$

Proof. ($\Rightarrow$) If S is a basis, then by definition it spans V, so such λ_i exist. Suppose $v = \sum \lambda_i e_i = \sum \mu_i e_i$. Then $\sum(\lambda_i - \mu_i)e_i = 0$. Linear independence implies $\lambda_i - \mu_i = 0$ for all i. Hence uniqueness.

($\Leftarrow$) If every v has a unique representation, then:

- Existence implies S spans V.
- Uniqueness implies S is linearly independent.

Thus S is a basis. $\square$

Submodule

Definition.

Let R be a CRWU, V an R–module. A subset $W \subseteq V$ is called a **submodule** of V if:

1. $(W, +)$ is a subgroup of $(V, +)$,
2. $\forall r \in R,\ w \in W : r * w \in W$.

Equivalently, W is a submodule iff

$$\forall \lambda, \mu \in R,\ v, w \in W, \quad \lambda v + \mu w \in W.$$

Null Space and Image as Submodules

Let $A \in M_{m\times n}(R)$.

1. Null Space:

$$\text{Null}(A) = \{x \in R^n \mid Ax = 0_{R^m}\}.$$

Proof: If $x, y \in \text{Null}(A)$ and $\lambda, \mu \in R$,

$$A(\lambda x + \mu y) = \lambda Ax + \mu Ay = 0.$$

Hence closed under addition and scalar multiplication → submodule of R^n.

2. Image:

$$\text{Im}(A) = \{Ax \mid x \in R^n\} \subseteq R^m.$$

If $y_1 = Ax_1$, $y_2 = Ax_2$, then for any $\lambda, \mu \in R$,

$$\lambda y_1 + \mu y_2 = A(\lambda x_1 + \mu x_2) \in \text{Im}(A).$$

Hence $\text{Im}(A)$ is a submodule of R^m. □

Example with Numerical Matrix

Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & -4 & -8 \end{bmatrix} \in M_{2\times 3}(\mathbb{Q}).$$

Find Null(A):

$$Ax = 0 \Rightarrow \begin{cases} x_1 + 2x_2 + 3x_3 = 0, \\ -2x_1 - 4x_2 - 8x_3 = 0. \end{cases}$$

Second equation is $-2\times$ first, redundant. Solve first:

$$x_1 = -2x_2 - 3x_3.$$

Hence

$$x = (x_1, x_2, x_3) = x_2(-2, 1, 0) + x_3(-3, 0, 1).$$

Therefore

$$\mathrm{Null}(A) = \mathrm{Span}_{\mathbb{Q}}\{(-2, 1, 0), (-3, 0, 1)\}.$$

Image of the Same Matrix (continued)

$$\mathrm{Im}(A) = \mathrm{Span}_{\mathbb{Q}} \left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \end{bmatrix}, \begin{bmatrix} 3 \\ -8 \end{bmatrix} \right\}.$$

Notice that

$$\begin{bmatrix} 2 \\ -4 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \qquad \begin{bmatrix} 3 \\ -8 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 0 \\ -2 \end{bmatrix},$$

so only one of these is independent. Therefore,

$$\mathrm{Im}(A) = \mathrm{Span}_{\mathbb{Q}} \left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}.$$

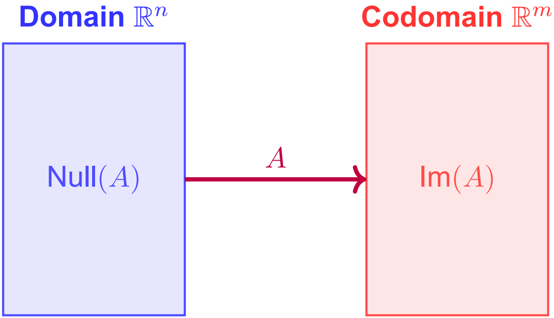
Hence

$$\dim_{\mathbb{Q}}(\mathrm{Im}(A)) = 1, \quad \dim_{\mathbb{Q}}(\mathrm{Null}(A)) = 2.$$

Observation (Rank–Nullity analogue):

$$\dim_{\mathbb{Q}}(\mathrm{Im}(A)) + \dim_{\mathbb{Q}}(\mathrm{Null}(A)) = 3 = \text{number of columns}.$$

Submodules as Linear Substructures



Interpretation:

- Null(A) lives in the domain R^n (vectors mapped to zero)
- Im(A) lives in the codomain R^m (all possible outputs)
- Each is a submodule closed under addition and scalar multiplication

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Remarks and Extensions

- Every vector space is a module, but not every module is a vector space (lack of division in R).
- Submodules play the same role as subspaces.
- The null and image submodules generalize the kernel and image of a linear map.
- Free modules generalize coordinate spaces R^n.
- If R is a principal ideal domain, many results from linear algebra (rank, basis, independence) extend naturally.

Important note: Modules over non-fields can have surprising behaviors — e.g., not every submodule has a complement, not every module has a basis.

Summary of Lecture 4

- Defined an R–module as an abelian group with a compatible scalar multiplication.
- Showed that for a CRWU, R^n, $M_{m\times n}(R)$, and $R[x]$ are R–modules.
- Introduced **linear independence**, **span**, and **basis** in modules.
- Proved characterization: S is a basis $\iff$ each v has a unique expression.
- Defined submodules and proved closure criterion $\lambda v + \mu w \in W$.
- Proved that Null(A) and Im(A) are submodules.
- Worked out complete example for $A \in M_{2\times 3}(\mathbb{Q})$, computing both Null$(A)$ and Im(A).

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Thanks