The University of Manchester

Combinatorial Mesh Calculus (CMC): Lecture 1

Lectured by: Dr. Kiprian Berbatov[1]

Lecture Notes Compiled by: Muhammad Azeem[1]

Under the supervision of: Prof. Andrey P. Jivkov[1]

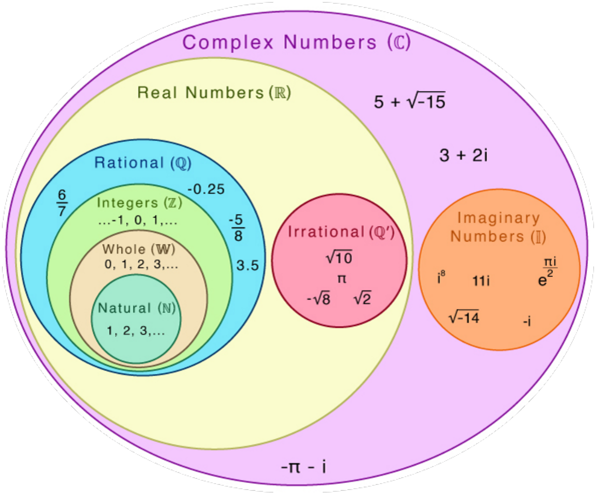
[1] Department of Mechanical and Aerospace Engineering, The University of Manchester, Oxford Road, Manchester M13 9PL, UK

Basic Definitions

The University of Manchester

- **Natural numbers:** $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ or $\{1, 2, 3, \dots\}$, choice depends on convention.
- **Positive naturals:** $\mathbb{N}^+ = \{1, 2, 3, \dots\}$
- **Integers:** $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$
- **Positive integers:** $\mathbb{Z}^+ = \{1, 2, 3, \dots\}$
- **Rational numbers:** $\mathbb{Q} = \left\{\frac{p}{q} \mid p \in \mathbb{Z},\ q \in \mathbb{Z} \setminus \{0\}\right\}$
- **Positive rationals:** $\mathbb{Q}^+ = \{q \in \mathbb{Q} \mid q > 0\}$
- **Real numbers:** $\mathbb{R} = \{x \mid -\infty < x < \infty\}$
- **Positive reals:** $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$

Basic Definitions



The University of Manchester

Set-Builder / Predicate Notation

Let a property $\varphi(x)$ and a set A

$$B = \{\, x \in A \mid \varphi(x) \text{ is true}\}$$

means "the subset of A whose members satisfy property φ."

Examples:

- Positive rational numbers can be written like

$$\mathbb{Q}^+ = \{x \in \mathbb{Q} \mid x > 0\}, \qquad \mathbb{N} = \{n \in \mathbb{Z} \mid n \geq 0\}$$

- If X is a set and $n \in \mathbb{N}$, define

$$X^n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in X \text{ for } i = 1, \ldots, n\}.$$

- If $m, n \in \mathbb{N}$, define the set of $m \times n$ matrices over X:

$$M_{m\times n}(X) = \{\, [a_{ij}] \mid a_{ij} \in X,\ 1 \leq i \leq m,\ 1 \leq j \leq n\}\,.$$

Logical Connectives: Basic Laws

Let φ and ψ be logical statements (predicates). We have:

$$
\begin{aligned}
\neg(\neg\varphi) \equiv \varphi \quad & \text{(negation)},\\
\varphi \wedge \psi \quad & \text{(conjunction/logical AND)},\\
\varphi \vee \psi \quad & \text{(disjunction/logical OR)},\\
\varphi \Rightarrow \psi \quad & \text{(implication)},\\
\varphi \iff \psi \quad & \text{(bi-conditional, equivalence)}.
\end{aligned}
$$

Some equivalences:

$$\varphi \Rightarrow \psi \;\equiv\; \neg\varphi \vee \psi, \quad \neg(\varphi \wedge \psi) \equiv \neg\varphi \vee \neg\psi \quad \text{(De Morgan)}, \ldots$$

The University of Manchester

Quantifiers: $\forall$, $\exists$

Universal quantifier:

$\forall x \in A,\ \varphi(x)$: for all x in $A, \varphi(x)$ holds.

Existential quantifier:

$\exists x \in A,\ \varphi(x)$: there is at least one $x \in A$ with $\varphi(x)$.

Negation rules:

$$\neg\big(\forall x \in A\, \varphi(x)\big) \equiv \exists x \in A\, \neg\varphi(x),$$

$$\neg\big(\exists x \in A\, \varphi(x)\big) \equiv \forall x \in A\, \neg\varphi(x).$$

Continuity and Uniform Continuity

Continuity

Let $I \subset \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$. We say f is continuous if

$$\forall x \in I,\ \forall \varepsilon > 0,\ \exists \delta > 0,\ \forall y \in I,\ |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon.$$

Uniform continuity

f is uniformly continuous if

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall x, y \in I,\ |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon.$$

Remark:

Uniform continuity is a stronger (global) condition: δ must work for all x, not depend on x.

Sets

Let A, B be sets.

Then

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\},$$
$$A \cap B = \{x \mid x \in A \text{ and } x \in B\},$$
$$A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}.$$

Power set:

Boolean algebra of subsets

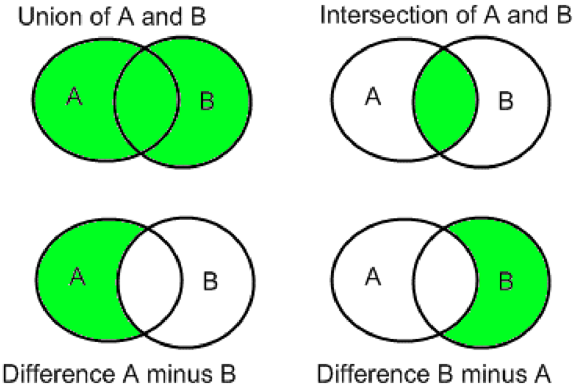
$$\mathcal{P}(A) = \{\, S \mid S \subseteq A \,\} \qquad |\mathcal{P}(A)| = 2^{|A|}$$

Sets

Remarks:

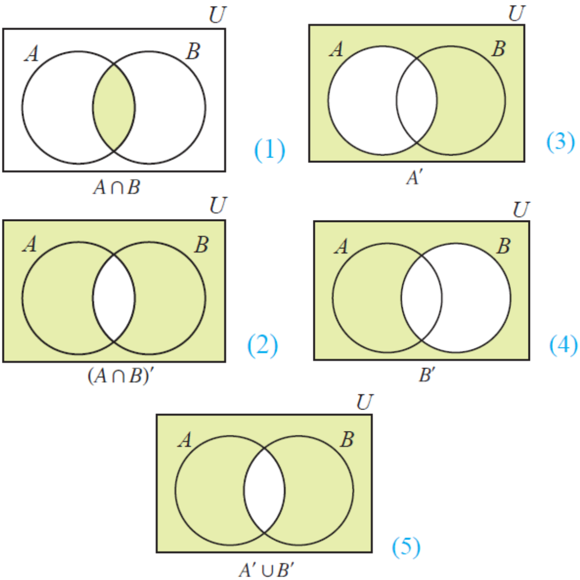
- Each element of $\mathcal{P}(A)$ is a subset of A (including $\emptyset$ and A itself).
- If A has n elements, then $\mathcal{P}(A)$ has 2^n elements.
- Equipped with the operations of union ($\cup$), intersection ($\cap$), and complement (^{c}), $\mathcal{P}(A)$ forms a **Boolean algebra**.
- The smallest element (zero) is $\emptyset$, and the greatest element (unity) is A.

Visualisation



Venn diagrams to illustrate union, intersection, difference.

Visualisation



Ordered Pairs and Cartesian Product

Definition:

The ordered pair (a, b) is defined by the Kuratowski construction:

$$(a, b) = \{\{a\}, \{a, b\}\}.$$

One checks $(a, b) = (a', b')$ iff $a = a'$ and $b = b'$.

Cartesian product:

$$A \times B = \{(a, b) \mid a \in A,\ b \in B\}.$$

Cardinality (finite case): If $|A| = m, |B| = n$, then

$$|A \times B| = m \cdot n.$$

More generally, for X^n, $|X^n| = |X|^n$ for finite X.

Function and Equality of Functions

Let X, Y be sets. A *function* $f : X \to Y$ ($X \xrightarrow{f} Y$) is a subset

$$f \subseteq X \times Y$$

such that

- For every $x \in X$, there is a unique $y \in Y$ with $(x, y) \in f$.

Definition

Two functions $f, g : X \to Y$ are *equal*, $f = g$, if

$$\forall x \in X,\ f(x) = g(x).$$

Example:

$\sin^2 x + \cos^2 x = 1$ defines a function identically equal to constant function $1(x) = 1$, for domain $\mathbb{R}$.

Composition of Functions

If $f : X \to Y$ and $g : Y \to Z$, define

$$g \circ f : X \to Z, \quad (g \circ f)(x) = g(f(x)).$$

Associativity: If $h : Z \to W$, then

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

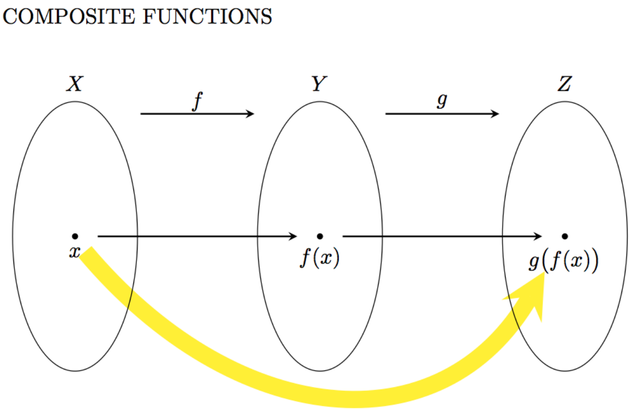
Example:

Let $X = \{a, b, c\},\ Y = \{0, 1\},\ Z = \{w, x, y, z\}$. Define

$$f(a) = 0,\ f(b) = 1,\ f(c) = 0, \quad g(0) = w,\ g(1) = z.$$

Then $g \circ f$ maps $a \mapsto w$, $b \mapsto z$, $c \mapsto w$. You can draw arrow diagrams: $X \xrightarrow{f} Y \xrightarrow{g} Z$

Visualization



Axiom of Set Equality:

Axiom (Extensionality). Let X and Y be two sets. Then

$$X = Y \iff (X \subseteq Y \text{ and } Y \subseteq X),$$

that is,

$$X = Y \iff \big(\forall x\,(x \in X \Rightarrow x \in Y)\big) \text{ and } \big(\forall x\,(x \in Y \Rightarrow x \in X)\big).$$

Interpretation: Two sets are equal if and only if they contain exactly the same elements.

Example: Let $A = \{1, 2, 3\}$ and $B = \{x \in \mathbb{N} \mid x < 4\}$.

Proof.

(1) If $x \in A$, then $x \in \{1, 2, 3\}$, hence $x < 4$ and $x \in B$. Thus $A \subseteq B$.

(2) If $x \in B$, then $x < 4$ and $x \in \mathbb{N}$, so $x \in \{1, 2, 3\} = A$. Hence $B \subseteq A$.

By the Axiom of Extensionality, $A = B$.

Identity Function and Its Property

Definition:

For any set X, the identity function is

$$\mathrm{id}_X : X \to X, \quad \mathrm{id}_X(x) = x \quad \forall x \in X.$$

Proposition:

For any function $f : X \to Y$,
$f \circ \mathrm{id}_X = f, \qquad \mathrm{id}_Y \circ f = f.$

Proof.

Let $x \in X$. By definition of composition:
$(f \circ \mathrm{id}_X)(x) = f(\mathrm{id}_X(x)) = f(x)$, since $\mathrm{id}_X(x) = x$. Hence
$f \circ \mathrm{id}_X = f$.
Similarly, for the right composition:
$(\mathrm{id}_Y \circ f)(x) = \mathrm{id}_Y(f(x)) = f(x)$, since $\mathrm{id}_Y(y) = y$ for all $y \in Y$.
Thus $\mathrm{id}_Y \circ f = f$. $\square$ $\square$

Injective, Surjective, Bijective

Let $f : X \to Y$.

- f is **injective** (one-to-one) if

$$\forall x_1, x_2 \in X,\ f(x_1) = f(x_2) \Rightarrow x_1 = x_2.$$

- f is **surjective** (onto) if

$$\forall y \in Y,\ \exists x \in X,\ f(x) = y.$$

- f is **bijective** if it is both injective and surjective.

Examples:

- id_X is bijective.
- $f(x) = x^2$ on $\mathbb{R} \to \mathbb{R}$ is not injective (two preimages);
- restrict to $[0, \infty)$, then it becomes injective (and bijective onto $[0, \infty)$).

Left Inverse, Right Inverse, and Relation

Let $f : X \to Y$.

- A **left inverse** of f is a function $g : Y \to X$ such that

$$g \circ f = \mathrm{id}_X.$$

- A **right inverse** of f is a function $h : Y \to X$ such that

$$f \circ h = \mathrm{id}_Y.$$

- If a function has both a left inverse and a right inverse, they coincide and that common map is called the *inverse* f^{-1}.

Proposition:

1. f has a left inverse $\iff$ f is injective.
2. f has a right inverse $\iff$ f is surjective.
3. f is bijective $\iff$ f has a two-sided inverse.

Proofs: ???

Proof

(1) Left inverse $\Leftrightarrow$ Injective.

($\Rightarrow$) Assume there exists $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$. Let $f(x_1) = f(x_2)$. Applying g to both sides gives

$$g(f(x_1)) = g(f(x_2)) \Rightarrow \mathrm{id}_X(x_1) = \mathrm{id}_X(x_2) \Rightarrow x_1 = x_2.$$

Thus f is injective.

($\Leftarrow$) Assume f is injective. For each $y \in \mathrm{im}(f)$, there exists a unique $x \in X$ such that $f(x) = y$. Define $g : Y \to X$ by

$$g(y) = \begin{cases} x, & \text{if } y = f(x) \text{ for some } x \in X, \\ x_0, & \text{if } y \notin \mathrm{im}(f), \end{cases}$$

where $x_0 \in X$ is fixed arbitrarily. Then for all $x \in X$, $g(f(x)) = x$, hence $g \circ f = \mathrm{id}_X$. Therefore, f has a left inverse.

Proof

(2) Right inverse $\Leftrightarrow$ Surjective.

($\Rightarrow$) Assume there exists $h : Y \to X$ such that $f \circ h = \mathrm{id}_Y$. For every $y \in Y$,

$$f(h(y)) = y,$$

so y is an image of f. Hence f is surjective.

($\Leftarrow$) Assume f is surjective. Then for each $y \in Y$ there exists at least one $x \in X$ such that $f(x) = y$. Choose one such x (using the Axiom of Choice if necessary), and define $h(y) = x$. Then $f(h(y)) = y$ for all $y \in Y$, i.e. $f \circ h = \mathrm{id}_Y$. Thus h is a right inverse of f.

Proof

(3) Bijective $\Leftrightarrow$ Two–sided inverse.

($\Rightarrow$) If f is bijective, it is both injective and surjective. From (1) and (2) there exist $g, h : Y \to X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ h = \mathrm{id}_Y$. But for a bijection, these must coincide ($g = h = f^{-1}$). Hence f^{-1} satisfies both identities:

$$f^{-1} \circ f = \mathrm{id}_X, \qquad f \circ f^{-1} = \mathrm{id}_Y.$$

($\Leftarrow$) If a function f has a two–sided inverse f^{-1}, then $f^{-1} \circ f = \mathrm{id}_X$ (injectivity) and $f \circ f^{-1} = \mathrm{id}_Y$ (surjectivity).
Therefore, f is bijective. $\square$

MANCHESTER 1824
The University of Manchester

Uniqueness of Inverse and Composition

Uniqueness of Inverse

If f has both left inverse g and right inverse h, then one shows $g = h$. Thus the two-sided inverse is unique.

Theorem:

If $f : X \to Y$ and $g : Y \to Z$ are bijections, then $g \circ f$ is bijective and

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}.$$

Proof: ???

The University of Manchester

Proof

Since f and g are bijections, each has an inverse function $f^{-1} : Y \to X$ and $g^{-1} : Z \to Y$ satisfying:

$$f^{-1} \circ f = \mathrm{id}_X, \quad f \circ f^{-1} = \mathrm{id}_Y, \quad g^{-1} \circ g = \mathrm{id}_Y, \quad g \circ g^{-1} = \mathrm{id}_Z.$$

Step 1: Show that $g \circ f$ is bijective.
Injectivity: Suppose $(g \circ f)(x_1) = (g \circ f)(x_2)$. Then $g(f(x_1)) = g(f(x_2))$. Since g is injective, it follows that $f(x_1) = f(x_2)$. Applying injectivity of f, we get $x_1 = x_2$. Hence $g \circ f$ is injective.
Surjectivity: Let $z \in Z$. Since g is surjective, there exists $y \in Y$ such that $g(y) = z$. Since f is surjective, there exists $x \in X$ such that $f(x) = y$. Then

$$(g \circ f)(x) = g(f(x)) = g(y) = z.$$

Thus every $z \in Z$ has a preimage in X; therefore, $g \circ f$ is surjective.
Hence $g \circ f$ is bijective.

Proof

Step 2: Compute the inverse. We claim that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. To verify this, check both compositions:

(a) Left composition:

$$(f^{-1} \circ g^{-1}) \circ (g \circ f) = f^{-1} \circ (g^{-1} \circ g) \circ f = f^{-1} \circ \mathrm{id}_Y \circ f = f^{-1} \circ f = \mathrm{id}_X.$$

(b) Right composition:

$$(g \circ f) \circ (f^{-1} \circ g^{-1}) = g \circ (f \circ f^{-1}) \circ g^{-1} = g \circ \mathrm{id}_Y \circ g^{-1} = g \circ g^{-1} = \mathrm{id}_Z.$$

Both compositions give the identity maps on X and Z, respectively. Thus $(f^{-1} \circ g^{-1})$ is indeed the inverse of $(g \circ f)$.

Conclusion:

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}. \quad \square$$

MANCHESTER 1824
The University of Manchester

Bundles, Projection, and Sections

Definition:

Let E and M be sets, and let $\pi : E \to M$ be a surjection. We call (E, π, M) a *bundle over* M, and π the *projection map*.

Definition:

A *section* is a (right) inverse map $s : M \to E$ such that

$$\pi \circ s = \mathrm{id}_M .$$

Thus s "picks a point in each fiber" consistently.

Bundles, Projection, and Sections

Definition:

For $x \in M$, define the fiber

$$E_x = \pi^{-1}(x) = \{e \in E \mid \pi(e) = x\}.$$

Then E is the disjoint union of its fibers:

$$E = \bigsqcup_{x \in M} E_x.$$

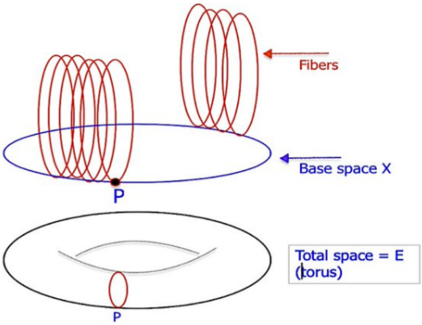
(Here $\bigsqcup$ means disjoint union.)

Visualization

Math rep. of a fiber bundle

$$\begin{array}{c} E \\ \Big\downarrow \pi \\ X \end{array}$$

Picture of a fiber bundle



MANCHESTER 1824
The University of Manchester

Example: Annulus as a Bundle

Take real numbers $0 \leq r_0 < r_1$. Define

$$E = \{(x, y) \in \mathbb{R}^2 \mid r_0 \leq x^2 + y^2 \leq r_1\}, \quad M = [r_0, r_1],$$

and the projection

$$\pi : E \to M,\ \pi(x, y) = \sqrt{x^2 + y^2}.$$

Then (E, π, M) is a bundle. The fiber over $r \in [r_0, r_1]$ is

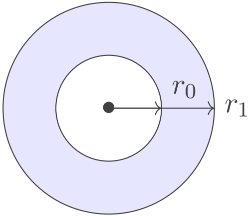
$$E_r = \{(x, y) \mid x^2 + y^2 = r^2\},$$

i.e. the circle of radius r. The total space is an annulus. This is an example of an I-bundle (interval-bundle) over a circle base.
A section would pick one point on each circle: e.g. define

$$s(r) = (r, 0),$$

then $\pi(s(r)) = r$.

Visualization



The University of Manchester

Proposition

A function $f : X \to Y$ is **bijective** if and only if it has a **two–sided inverse**; that is,

$$f \text{ is bijective} \iff \exists\, g : Y \to X \text{ such that } g \circ f = \mathrm{id}_X \text{ and } f \circ g = \mathrm{id}_Y .$$

Proof (⇒) Assume f is bijective. Then f is injective and surjective.

Existence of inverse function: For each $y \in Y$, surjectivity ensures the existence of at least one $x \in X$ such that $f(x) = y$. Injectivity guarantees that this x is unique. Hence, we can define a well–defined function $g : Y \to X$ by assigning $g(y) = x$, where $f(x) = y$.

Now, for all $x \in X$:

$$(g \circ f)(x) = g(f(x)) = x,$$

and for all $y \in Y$:

Proof

$$(f \circ g)(y) = f(g(y)) = y.$$

Thus $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$; g is a two–sided inverse of f.

(⇐) Conversely, suppose there exists $g : Y \to X$ such that

$$g \circ f = \mathrm{id}_X, \quad f \circ g = \mathrm{id}_Y.$$

Injectivity: If $f(x_1) = f(x_2)$, apply g to both sides:

$$g(f(x_1)) = g(f(x_2)) \Rightarrow \mathrm{id}_X(x_1) = \mathrm{id}_X(x_2) \Rightarrow x_1 = x_2.$$

Hence f is injective.

Proof

Surjectivity: For any $y \in Y$, set $x = g(y)$. Then

$$f(x) = f(g(y)) = (f \circ g)(y) = \mathrm{id}_Y(y) = y.$$

Therefore every $y \in Y$ has a preimage, so f is surjective.
Since f is both injective and surjective, f is bijective. $\square$

Notes

- In the bundle context, a section gives a right inverse of π.
- If π also had a left inverse, that would force π to be injective, which bundle projections typically are not (because fibers often contain multiple points).
- Thus for a general bundle, π is surjective but not injective, so it has sections, but no inverse redefining π as bijection.

MANCHESTER 1824
The University of Manchester

Summary and Roadmap

- We have defined standard sets ($\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$) and positive subsets.
- We introduced predicate/set-builder notation, Cartesian products, and matrices.
- Reviewed logical connectives and quantifiers; gave continuity, uniform continuity.
- Covered set operations (union, intersection, difference, power set) and the Cartesian product.
- Defined functions, composition, identity, and equality of functions.
- Introduced injectivity, surjectivity, bijectivity, and the connection to inverses (left, right).
- Introduced bundles, projections, fibers, and sections, with a concrete annulus example.

Homework

Prove or disprove:

1. f injective $\iff$ f has a left inverse.
2. f has right inverse $\Rightarrow f$ surjective.
3. f surjective $\Rightarrow f$ has right inverse.
4. f bijective $\iff$ f invertible.
5. Decomposition into fibers is indeed a disjoint union:
   $E = \bigsqcup_{x \in M} E_x$ is a disjoint decomposition into fibers.
   If $\pi : E \to M$ is a surjection, then $E = \bigsqcup_{x \in M} \pi^{-1}(x)$ and for $x \neq y$, $\pi^{-1}(x) \cap \pi^{-1}(y) = \emptyset$.

MANCHESTER
1824
The University of Manchester

Thanks